Intrinsic curvature

Now we want a relationship between 3D + 4D Riemann tensors.

Recall $\nabla_{[a}\nabla_{b]}V^c = \frac{1}{2}\,{}^{(4)}R^c{}_{dab}V^d$

So define 3d Riemann by $\boxed{D_{[a}D_{b]}V^c = \frac{1}{2}\,{}^{(3)}R^c{}_{dab}V^d \quad \text{For spatial } V}$

First write

$$D_bV^c = \gamma_e{}^c\gamma_b{}^d\nabla_dV^e \qquad \text{where we will assume } V^cn_c = 0 \text{ spatial}$$

$$= \gamma_b{}^d\left(\nabla_dV^c + n_en^c\nabla_dV^e\right)$$

But $K_{bd}V^dn^c = -\gamma_b{}^e\gamma_d{}^aV^dn^c\nabla_en_a$

$$= -\gamma_b{}^eV^an^c\nabla_en_a \underset{(V^an_a=0)}{=} \gamma_b{}^en^cn_a\nabla_eV^a$$

$$\Rightarrow \boxed{D_bV^c = \gamma^a{}_b\nabla_aV^c + K_{ba}V^an^c}$$

Now

$$D_a D_b V^c = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e \left[D_d V^g \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \left[\nabla_e(\gamma^F{}_d) \nabla_F V^g + (\nabla_e \nabla_F V^g)\gamma^F_d + K_{dF} V^F \nabla_e n^g \right.$$

$$\left. + K_{dF} (\nabla_e V^F) \cancelto{0}{n^g} + \cancelto{0}{V^F n^g} \nabla_e K_{dF} \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \left[\nabla_F V^g \nabla_e (n^F n_d) + (\nabla_e \nabla_F V^g) \delta_d{}^F + K_{dF} V^F \nabla_e n^g \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e \nabla_d V^g + K_{bF} V^F \underbrace{\gamma_a{}^e \gamma_g{}^c \nabla_e n^g}_{-K_a{}^c} + (\nabla_F V^g) n^F \underbrace{\gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e n_d}_{-\gamma_g{}^c K_{ab}}$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c D_e D_d V^g - K_a{}^c K_{bF} V^F - K_{ab} \gamma_g{}^c (\nabla_F V^g) n^F$$

Now $D_{[a} D_{b]} V^c = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \underbrace{\nabla_{[e} \nabla_{d]} V^g}_{\frac{1}{2} {}^{(4)}R^g{}_{Fed} V^F} - K^c{}_{[a} K_{b]F} V^F - \cancelto{0}{K_{[ab]}} \gamma_g{}^c (\nabla_F V^g) n^F$

So $\frac{1}{2} {}^{(3)}R^c{}_{Fab} V^F = \frac{1}{2} \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c\, {}^{(4)}R^g{}_{Fed} V^F - K^c{}_{[a} K_{b]F} V^F$

$$\Rightarrow \boxed{{}^{(3)}R^c{}_{Fab} = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c\, {}^{(4)}R^g{}_{Fed} - 2K^c{}_{[a} K_{b]F}}$$

or $\boxed{{}^{(3)}R_{abcd} = \gamma_a{}^e \gamma_b{}^F \gamma_c{}^g \gamma_d{}^h\, {}^{(4)}R_{efgh} - 2K_{b[d} K_{c]a}}$ $\boxed{\text{Gauss' Equation}}$

Gauss: Full spatial projection of ${}^{(4)}R_{abcd}$

What about one index of ${}^{(4)}R_{abcd}$ projected into normal direction?

Consider $D_a K_{bc} = \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \nabla_d K_{ef}$

$$= -\gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \, \nabla_d \left(\nabla_e n_f + n_e \overbrace{n^h \nabla_h n_f}^{a_f} \right)$$

$$= -\gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \left[\nabla_d \nabla_e n_f + \cancelto{0}{n_e \nabla_d a_f} + a_f \nabla_d n_e \right]$$

$(K_{ab} = \gamma_a{}^e \gamma_b{}^f \nabla_e n_f)$

$$= -\gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \left[\nabla_d \nabla_e n_f \right] + a_c K_{ab}$$

Antisymmetrize: $D_{[a} K_{b]c} = -\gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \left[\nabla_{[d} \nabla_{e]} n_f \right] + a_c \cancelto{0}{K_{[ab]}}$

$$= -\frac{1}{2} {}^{(4)}R_{def}{}^h n_h \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f$$

$$\Rightarrow \boxed{2 D_{[a} K_{b]c} = \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \, {}^{(4)}R_{edfg} n^g}$$

Codazzi eqn.

Gauss-Codazzi are conditions for embedding slice with (γ_{ab}, K_{ab}) in 4D manifold with g_{ab}.

Can now write constraint parts of $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

Now contract Gauss w/ γ^{ac}:

$$ {}^{(3)}R^{a}{}_{bad} = {}^{(3)}R_{bd} = \gamma^{eg}\gamma_b{}^f\gamma_d{}^h\,{}^{(4)}R_{efgh} - K_{bd}\underbrace{K}_{K^c{}_c} + K_b{}^a K_{da} $$

$$ \Rightarrow {}^{(3)}R_{bd} = \gamma^{eg}\gamma_b{}^f\gamma_d{}^h\,{}^{(4)}R_{efgh} - KK_{bd} + K_{da}K^a{}_b $$

contract again $\quad {}^{(3)}R = \gamma^{eg}\gamma^{fh}\,{}^{(4)}R_{efgh} - K^2 + K^a{}_b K^b{}_a$

$$ \text{Now } \gamma^{eg}\gamma^{fh}\,{}^{(4)}R_{efgh} = \overbrace{g^{eg}g^{fh}\,{}^{(4)}R_{efgh}}^{{}^{(4)}R} + \cancelto{0}{n^e n^g n^f n^h\,{}^{(4)}R_{efgh}} $$

$$ + \underbrace{g^{eg}n^f n^h\,{}^{(4)}R_{efgh}}_{{}^{(4)}R_{fh}n^f n^h} + \underbrace{n^e n^g \gamma^{fh}\,{}^{(4)}R_{efgh}}_{{}^{(4)}R_{eg}n^e n^g} $$

$$ = {}^{(4)}R + 2\,{}^{(4)}R_{ab}n^a n^b $$

but ${}^{(4)}R_{ab} = {}^{(4)}G_{ab} + \frac{1}{2}{}^{(4)}R g_{ab}$ $\quad \Rightarrow \gamma^{eg}\gamma^{fh}\,{}^{(4)}R_{efgh} = \cancel{{}^{(4)}R} + 2\,{}^{(4)}G_{ab}n^a n^b$

$$ + \cancel{{}^{(4)}R\, g_{ab}n^a n^b} $$

$$ = 2\,{}^{(4)}G_{ab}n^a n^b $$

$$ = 2T_{ab}n^a n^b $$

let $\rho \equiv T_{ab}n^a n^b$ $\quad$ energy density seen by normal observers

$$ \Rightarrow \boxed{{}^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi\rho} $$

Hamiltonian constraint

Now contract Codazzi:

$$\gamma^{bc}\, 2 D_{[a} K_{b]c} = \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \gamma^{bc}\, {}^{(4)}R_{edfg} n^g$$

$$D_a K - D_b K^b{}_a = {}^{(4)}R_{edfg} n^g \gamma^{ef} \gamma_a{}^d$$

$$= \gamma_a{}^d \left(g^{ef} + \cancelto{0}{n^e n^f}\right) {}^{(4)}R_{edfg} n^g$$

$$= \gamma_a{}^d n^g\, {}^{(4)}R_{dg}$$

$$= \gamma_a{}^d n^g \left[{}^{(4)}G_{dg} + \frac{1}{2}\, {}^{(4)}R\, \cancelto{0}{g_{dg}} \right]$$

$$= \gamma_a{}^d n^g\, {}^{(4)}G_{dg}$$

$$\equiv -8\pi S_a$$

$S_a \equiv \gamma_a{}^b n^c T_{bc}$

is momentum density seen by normal observers.

$$\Rightarrow \boxed{D_b K^b{}_a - D_a K = 8\pi S_a}$$

Momentum constraint!

Time evolution:

Look at $\mathcal{L}_{\vec{t}}\,\gamma_{ab}$, $\mathcal{L}_{\vec{t}}\,K_{ab}$.   drag metric, K_{ab} thru time

What is $\vec{t}$?   Need: $t^a t_a < 0$ timelike

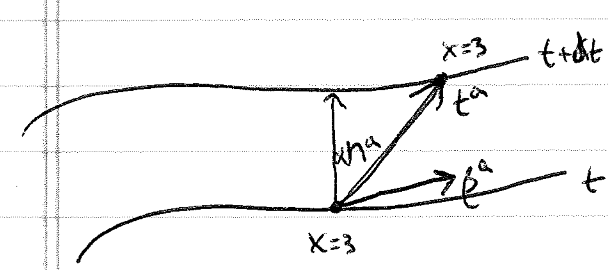
$t^a \Omega_a = 1$ t^a dual to filiation 1-form

Recall $\Omega_a = \nabla_a t$ or $\tilde{\Omega} = \tilde{d}t$ and $\Omega_a = \frac{-n_a}{\alpha}$ lapse function

Most general: $\boxed{t^a = \alpha n^a + \beta^a}$ arbitrary spatial vector = "shift vector"

$n_a \beta^a = 0$

then $t^a \Omega_a = -\frac{1}{\alpha} t_a n^a = -\frac{1}{\alpha}\left[\alpha n^a + \cancelto{0}{\beta^a}\right] n_a = 1$



$d\tau = \alpha\, dt$

First look at $\mathcal{L}_{\vec{t}}\,\gamma_{ab} = \mathcal{L}_{\alpha n+\beta}\,\gamma_{ab}$

$$= \mathcal{L}_{\alpha n}\gamma_{ab} + \mathcal{L}_{\beta}\gamma_{ab}$$

$$= \alpha n^c \nabla_c \gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}(\alpha n^c) + \mathcal{L}_{\beta}\gamma_{ab}$$

$$= \alpha\Big[\underbrace{n^c\nabla_c\gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}n^c}_{\mathcal{L}_n\gamma_{ab}\ =\ -2K_{ab}}\Big] + 2\cancelto{0}{\big[\gamma_{c(a}\nabla_{b)}\alpha\big]n^c} + \mathcal{L}_{\beta}\gamma_{ab}$$

$$\Rightarrow \quad \boxed{\mathcal{L}_{\vec{t}}\,\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_{\beta}\gamma_{ab}}$$

$$\mathcal{L}_{\alpha n}\gamma_{ab} = -2\alpha K_{ab}$$

Now show $\mathcal{L}_{\alpha n}[\text{spatial}] = [\text{spatial}]$ good enough to show $\mathcal{L}_{\alpha n}\gamma^a{}_b = 0$

$$\mathcal{L}_{\alpha n}\gamma^a{}_b = \alpha n^c \nabla_c \gamma^a{}_b + \gamma_c{}^a \nabla_b(\alpha n^c) - \gamma_b{}^c \nabla_c(\alpha n^a)$$

$$= \alpha\left[n^c\nabla_c\gamma^a{}_b + \gamma_c{}^a\nabla_b n^c - \gamma_b{}^c\nabla_c n^a\right] + \cancelto{0}{n^c\gamma_c{}^a}\nabla_b\alpha - n^a\gamma_b{}^c\nabla_c\alpha$$

$$= \alpha\left[n^c\nabla_c(n^a n_b) + \gamma_c{}^a\nabla_b n^c - \gamma_b{}^c\nabla_c n^a\right] - n^a D_b\alpha$$

$$= \alpha\left[n^c n^a\nabla_c n_b + \underbrace{n^c n^b\nabla_c n^a - \gamma_b{}^c\nabla_c n^a}_{-\nabla_b n^a} + \underbrace{\gamma_c{}^a\nabla_b n^c}_{\underbrace{\delta_c{}^a\nabla_b n^c}_{\nabla_b n_a} + n_c \cancelto{0}{n^a\nabla_b n^c}}\right] - n^a D_b\alpha$$

$$0$$

$$= \alpha n^c n^a\nabla_c n_b - n^a D_b\alpha$$

$$= \alpha n^a a_b - n^a D_b\alpha$$

Lemma: $D_b\alpha = \alpha a_b$ so $\boxed{\mathcal{L}_{\alpha n}\gamma^a{}_b = 0}$

Pf of Lemma:

$$\nabla_b n_a = \nabla_b(-\alpha\Omega_a) = -\alpha\nabla_b\Omega_a - \Omega_a\nabla_b\alpha$$

$$= -\alpha\nabla_a\Omega_b - \Omega_a\nabla_b\alpha$$

(because $\nabla_{[a}\Omega_{b]} = 0$ i.e. $d\Omega = 0$)

$$= +\alpha\nabla_a\left(\frac{n_b}{\alpha}\right) + \frac{n_a}{\alpha}\nabla_b\alpha$$

$$= \nabla_a n_b - \frac{n_b\nabla_a\alpha}{\alpha} + \frac{n_a}{\alpha}\nabla_b\alpha$$

Now $0 = n^a\nabla_b n_a = n^a\nabla_a n_b - \frac{n^a n_b\nabla_a\alpha}{\alpha} - \frac{1}{\alpha}\nabla_b\alpha$

$$= a_b - \frac{1}{\alpha}\left[n^a n_b + \delta^a_b\right]\nabla_a\alpha$$

$$= a_b - \frac{1}{\alpha}D_b\alpha \quad \boxtimes$$

$$\boxed{a_b = D_b \ln\alpha}$$