Abstract index notation: (Wald 1984)

- Abstract notation: $\tilde{\tilde{\tilde{T}}}(-,-,-\ldots)$ is a tensor

  $\vec{e}_\beta$ is a basis vector

  etc.

  Precise but — Tedious

- Component notation:

  choose basis, then $T^\alpha{}_{\beta\gamma} \equiv \tilde{\tilde{\tilde{T}}}(\tilde{\omega}^\alpha, \vec{e}_\beta, \vec{e}_\gamma)$

  Can write equations w/ components i.e. $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

  $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ — Need basis

- <u>Abstract</u> <u>index</u> notation:

  takes 1-form, takes vector, takes vector

  $T^a{}_{bc}$ <u>means</u> $\tilde{\tilde{\tilde{T}}}(\_, \_, \_)$

  $\nabla_a V_b$ <u>means</u> $\tilde{\tilde{\nabla}} V(-,-)$ cov. deriv.

  [Not $\nabla_{\vec{e}_a} V_b$]

  So we can write $t^a \nabla_a V^b = 0$ [$\vec{V}$ ∥ transported along $\vec{t}$]

  $(\nabla_a \nabla_b - \nabla_b \nabla_a) V_c = R_{abc}{}^d V_d$ [def. of Riemann]

  $G_{ab} = 8\pi T_{ab}$

  $\nabla^a T_{ab} = 0$ etc.

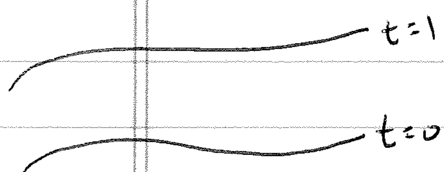
- Makes clear when eqs are tensor eqs rather than in specific basis

GR in initial value Formulation

Motivation: consider Maxwell

$$\nabla\cdot\mathbf{E} = 4\pi\rho \qquad \dot{\mathbf{E}} = \nabla\times\mathbf{B} - 4\pi\mathbf{J}$$

$$\underbrace{\nabla\cdot\mathbf{B} = 0}_{\text{constraints}} \qquad \underbrace{\dot{\mathbf{B}} = -\nabla\times\mathbf{E}}_{\text{evolution}}$$



Can set up problem at $t=0$ by solving constraints.

Get $t=1$ by solving ev. eqs.

Note: evolution eqs preserve constraints:

$$\nabla\cdot\dot{\mathbf{E}} = \cancelto{0}{\nabla\cdot\nabla\times\mathbf{B}} - 4\pi\nabla\cdot\mathbf{J} = -4\pi\dot{\rho} \quad\Rightarrow\quad \frac{\partial}{\partial t}\left[\nabla\cdot\mathbf{E} - 4\pi\rho\right] = 0$$

$$\nabla\cdot\dot{\mathbf{B}} = -\cancelto{0}{\nabla\cdot\nabla\cdot\mathbf{E}} \quad\Rightarrow\quad \frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) = 0$$

– Want to separate $G_{ab} = 8\pi T_{ab}$ in same way.

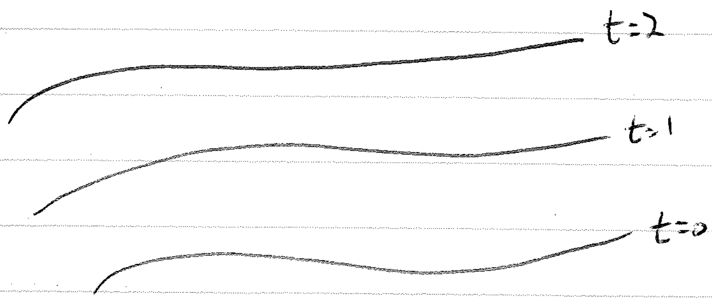
As it is, time/space all mixed up.

Assume closed 1-form $\tilde{\Omega}$ (closed means $\tilde{d}\tilde{\Omega} = 0$)

Foliating spacetime.

Can write $\tilde{\Omega} = \tilde{d}t$

or $\Omega_a = \nabla_a t$



don't choose coords yet, just slices.

Define $n_a \equiv -\alpha \Omega_a$ such that $n_a n^a = -1$

n_a = unit normal 1-form. α called "lapse function".

<u>assume $\alpha > 0$</u>

Note $\Omega_a \Omega^a = \dfrac{-1}{\alpha^2}$, $n_{[a}\nabla_b n_{c]} = 0$ hypersurface orthogonal

Define spatial metric $\boxed{\gamma_{ab} \equiv g_{ab} + n_a n_b}$

Metric induced on a 3D hypersurface. γ_{ab} is <u>spatial</u>

since $n^a \gamma_{ab} = n^a g_{ab} + n^a n_a n_b$
$= n_b - n_b = 0$

Also, $\gamma^a{}_b \equiv \delta^a{}_b + n^a n_b$ is a projection operator onto the slice.

So a tensor $\gamma^a{}_b T^{bcd\ldots}{}_{efg\ldots}$ is purely spatial on index a

since $n_a \left[\gamma^a{}_b T^{bcd\ldots}{}_{efg\ldots}\right] = \cancel{n_a \gamma^a{}_b}^{\,0} T^{bcd\ldots}{}_{efg\ldots} = 0$

$\Rightarrow$ Define 3D covariant derivative D

Must satisfy: — Spatial

— $D_a \gamma_{bc} = 0$ Cov deriv compatible w/ 3-metric

Define $D_a f \equiv \gamma_a{}^b \nabla_b f$

$D_a T^b{}_c \equiv \gamma^e_a \gamma^d_c \gamma^b_f \nabla_e T^f{}_d$ Project every index

etc.

<u>Check</u> $D_a \gamma_{bc} = \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \nabla_d (g_{ef} + n_e n_f)$

$= \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \left[\cancel{n_e}^{\,0} \nabla_d n_f + \cancel{n_f}^{\,0} \nabla_d n_e\right] = 0$

✓ ok

Now we want "time deriv" of metric - drag along normal.

Define $K_{ab} \equiv -\frac{1}{2} \mathcal{L}_n \gamma_{ab}$ (Lie deriv along n^a)

K_{ab} = "Extrinsic curvature"

= "2nd Fundamental Form"

$= -\frac{1}{2}\left[n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a n^c + \gamma_{ac} \nabla_b n^c\right]$

$K_{ab} = -\frac{1}{2}\left[n^c \nabla_c (\overset{g_{ab}+\ \to 0}{n_a n_b}) + (g_{cb} + \cancel{n_c n_b}) \nabla_a n^c + (g_{ac} + \cancel{n_a n_c}) \nabla_b n^c\right]$

$\left(n^c \nabla_b n_c = \frac{1}{2} \nabla_b n^c n_c = 0\right)$

So $K_{ab} = \frac{-1}{2}\left[n^c \nabla_c (n_a n_b) + \underbrace{\underbrace{2 g_{c(a} \nabla_{b)} n^c}_{2\nabla_{(b} g_{a)c} n^c}}_{2\nabla_{(b} n_{a)}}\right]$

$\Rightarrow K_{ab} = \frac{-1}{2}\left[n^c \nabla_c (n_a n_b) + 2\nabla_{(b} n_{a)}\right]$

Now consider $-\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)}$

$$= -\left(\delta_a{}^c \delta_b{}^d + \delta_a{}^c n_b n^d + n_a n^c \delta_b{}^d + \cancelto{0}{n_a n_b n^c n^d}\right) \nabla_{(c} n_{d)}$$

$$= -\left(\nabla_{(a} n_{b)} + \frac{1}{2} n_b n^d \nabla_d n_a + \frac{1}{2} n_a n^c \nabla_c n_b\right)$$

$$= -\left(\nabla_{(a} n_{b)} + \frac{1}{2} n^c \nabla_c (n_a n_b)\right)$$

So $\boxed{K_{ab} = -\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)}}$ spatial & symmetric

But look at irrotational condition $n_{[a} \nabla_b n_{c]} = 0$

dot into n^a $\Rightarrow 6 n^a n_{[a} \nabla_b n_{c]} = 0 = -2\nabla_{[b} n_{c]}$

$+ n^a \left[n_b \nabla_c \cancelto{0}{n_a} + n_c \nabla_a n_b - n_b \nabla_a n_c - n_c \cancelto{0}{\nabla_b n_a}\right]$

So $\gamma_d{}^c \gamma_e{}^b \nabla_{[c} n_{b]} = \frac{-1}{2} \gamma_d{}^c \gamma_e{}^b \left(n^a n_c \nabla_a n_b - n^a n_b \nabla_a n_c\right) = 0$

So $\boxed{K_{ab} = -\gamma_a^c \gamma_b^d \nabla_c n_d}$

Another relation: define acceleration of unit normal vector field

$$a_c \equiv n^a \nabla_a n_c$$

(Spatial because $n^c a_c = n^a n^c \nabla_a n_c = 0$)

Then $\nabla_a n_b = (\gamma_a{}^c - n^c n_a)(\gamma_b{}^d - n^d n_b) \nabla_c n_d$

$$= \gamma_a{}^c \gamma_b{}^d \nabla_c n_d - \gamma_a{}^c n^d \cancel{n_b \nabla_c n_d}^{\,0} - \gamma_b{}^d n^c n_a \nabla_c n_d$$

$$+ n_a n_b n^c \cancel{n^d \nabla_c n_d}^{\,0}$$

$$= -K_{ab} - n_a a_b$$

$$\Rightarrow \boxed{K_{ab} = -\nabla_a n_b - n_a a_b}$$

Intrinsic curvature

Now we want a relationship between 3D + 4D Riemann tensors.

Recall $\nabla_{[a}\nabla_{b]}V^c = \frac{1}{2}\,{}^{(4)}R^c{}_{dab}V^d$

So define 3d Riemann by
$$\boxed{D_{[a}D_{b]}V^c = \frac{1}{2}\,{}^{(3)}R^c{}_{dab}V^d \quad \text{For spatial } V}$$

First write

$$D_bV^c = \gamma_e{}^c\gamma_b{}^d\nabla_dV^e$$
where we will assume $V^cn_c = 0$ spatial

$$= \gamma_b{}^d\left(\nabla_dV^c + n_en^c\nabla_dV^e\right)$$

But $K_{bd}V^dn^c = -\gamma_b{}^e\gamma_d{}^aV^dn^c\nabla_en_a$

$$= -\gamma_b{}^eV^an^c\nabla_en_a \xrightarrow{(V^an_a=0)} = \gamma_b{}^en^cn_a\nabla_eV^a$$

$$\Rightarrow \boxed{D_bV^c = \gamma^a{}_b\nabla_aV^c + K_{ba}V^an^c}$$

Now

$$D_a D_b V^c = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e \left[D_d V^g \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \left[\nabla_e(\gamma^f{}_d) \nabla_f V^g + (\nabla_e \nabla_f V^g)\gamma^f_d + K_{df} V^f \nabla_e n^g \right.$$

$$\left. + K_{df} (\nabla_e V^f) \cancelto{0}{n^g} + \cancelto{0}{V^f n^g} \nabla_e K_{df} \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \left[\nabla_f V^g \nabla_e (n^f n_d) + (\nabla_e \nabla_f V^g) \delta_d{}^f + K_{df} V^f \nabla_e n^g \right]$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e \nabla_d V^g + K_{bf} V^f \underbrace{\gamma_a{}^e \gamma_g{}^c \nabla_e n^g}_{-K_a{}^c} + (\nabla_f V^g) n^f \underbrace{\gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e n_d}_{-\gamma_g{}^c K_{ab}}$$

$$= \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \nabla_e \nabla_d V^g - K_a{}^c K_{bf} V^f - K_{ab} \gamma_g{}^c (\nabla_f V^g) n^f$$

Now $$D_{[a} D_{b]} V^c = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \underbrace{\nabla_{[e} \nabla_{d]} V^g}_{\frac{1}{2} {}^{(4)}R^g{}_{fed} V^f} - K^c{}_{[a} K_{b]f} V^f - \cancelto{0}{K_{[ab]}} \gamma_g{}^c (\nabla_f V^g) n^f$$

So $$\frac{1}{2} {}^{(3)}R^c{}_{fab} V^f = \frac{1}{2} \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \, {}^{(4)}R^g{}_{fed} V^f - K^c{}_{[a} K_{b]f} V^f$$

$$\Rightarrow \boxed{{}^{(3)}R^c{}_{fab} = \gamma_a{}^e \gamma_b{}^d \gamma_g{}^c \, {}^{(4)}R^g{}_{fed} - 2K^c{}_{[a} K_{b]f}}$$

or $${}^{(3)}R_{abcd} = \gamma_a{}^e \gamma_b{}^f \gamma_c{}^g \gamma_d{}^h \, {}^{(4)}R_{efgh} - 2K_{b[d} K_{c]a}$$

Gauss' Equation