How do you get GWs in the first place? Sources

Assume $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ not necessarily small

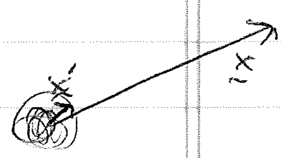
$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ $h \equiv h_{\alpha\beta}\eta^{\alpha\beta}$ not a tensor

demand $\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha} = 0$ Lorentz gauge

Then $G_{\mu\nu} = 8\pi T_{\mu\nu}$ $\Rightarrow$ $\bar{h}^{\mu\nu}{}_{,\alpha\beta}\eta^{\alpha\beta} = -16\pi\left(T^{\mu\nu} + t^{\mu\nu}\right)$

flat space deriv pseudotensor

Solution is $\boxed{\bar{h}^{\mu\nu}(t,x) = 4\int \frac{\left[T^{\mu\nu} + t^{\mu\nu}\right]_{ret}}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3x'}$



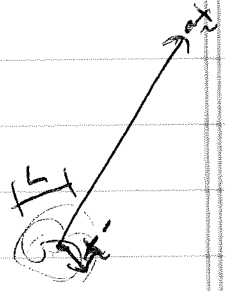
ret means evaluate at retarded time $t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'|$

still treating $\bar{h}^{\mu\nu}$ as field in flat spacetime.

Not true physically, but math ok. ($t^{\mu\nu}$ not unique)

Note: $\bar{h}^{\mu\nu}$ appears in $t^{\mu\nu}$, so not simple to solve.

We will make some approximations.



Slow-motion approximation

$L \equiv$ size of source

$\bar{\lambda} \equiv$ reduced wavelength $\qquad \lambda/2\pi$

Assume $L \ll \bar{\lambda}$

Then $|\underset{\sim}{x}| = r \gg L \gtrsim |\underset{\sim}{x}'|$ $\qquad$ so expand in $\underset{\sim}{x}'/r$

$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \sim \frac{1}{r} + \mathcal{O}(1/r^2)$$

$$T^{\mu\nu}(\underset{\sim}{x}', t-|\underset{\sim}{x}-\underset{\sim}{x}'|) = T^{\mu\nu}(\underset{\sim}{x}', t-r) + \underbrace{\underbrace{(r-|\underset{\sim}{x}-\underset{\sim}{x}'|)}_{\frac{x^j}{r}x'_j}\frac{\partial}{\partial t}T^{\mu\nu}(\underset{\sim}{x}, t-r)}_{\mathcal{O}(x'_j/\bar{\lambda}) \sim L/\bar{\lambda} \ll 1} + \ldots$$

$$\Rightarrow \boxed{\bar{h}^{\mu\nu}(t,\underset{\sim}{x}) = \frac{4}{r}\int\left[T^{\mu\nu}(\underset{\sim}{x}', t-r) + t^{\mu\nu}(\underset{\sim}{x}', t-r)\right] d^3x' + \ldots}$$

Want to find h_{ij}^{TT}

Can write $\int T^{ij}$ in terms of $\int T^{00}$

<u>how?</u>

Recall $(T^{\mu\nu} + t^{\mu\nu})_{,\nu} = 0$

So $(T^{0\nu}+t^{0\nu})_{,\nu} = 0 = (T^{00}+t^{00})_{,0} + (T^{0i}+t^{0i})_{,i}$

$\frac{\partial}{\partial t}$ of both sides $\Rightarrow$

$$(T^{00}+t^{00})_{,00} = -(T^{0i}+t^{0i})_{,i0}$$
$$= +(T^{ij}+t^{ij})_{,ij}$$

(using $(T^{i\nu}+t^{i\nu})_{,\nu}=0$)

Now consider $\left[x^i x^j (T^{00}+t^{00})\right]_{,00}$

$$= x^i x^j (T^{lm}+t^{lm})_{,lm}$$
$$= (x^i x^j (T^{lm}+t^{lm}))_{,lm} - 2\left[T^{ij}+t^{ij}\right]$$
$$-2\left[(T^{li}+t^{li})x^j + (T^{lj}+t^{lj})x^i\right]_{,l}$$

So

$$\int (T^{ij}+t^{ij})\, d^3x = \frac{1}{2}\frac{d^2}{dt^2}\int (T^{00}+t^{00}) x^i x^j\, d^3x$$

Gauss Law $(T+t \to 0$ far from source$)$

$$+ \cancel{\int \left[(T^{li}+t^{li})x^j + (T^{lj}+t^{lj})x^i\right]_{,l} d^3x} - \cancel{\frac{1}{2}\int \left[x^i x^j (T^{lm}+t^{lm})\right]_{,lm} d^3x}$$

$$\Rightarrow \bar{h}^{ij}(t, \underset{\sim}{x}) = \frac{4}{r}\frac{1}{2}\frac{d^2}{dt^2}\int \left[T^{00}(\underset{\sim}{x}', t-r) + t^{00}(\underset{\sim}{x}', t-r)\right] d^3x'$$

Now assume nearly Newtonian source:

$$t^{00} \sim (\Phi_{,j})^2 \sim M^2/L^4 \sim M/L \; T^{00} \quad << T^{00}$$

- density

Newtonian Grav potential

i.e. assume $M/L << 1$

Let $I_{jk}(t) \equiv \int T^{00}(\underset{\sim}{x},t)\, x_j x_k \, d^3x$

2nd moment "moment of inertia"

Then $$\boxed{\bar{h}^{ij}(x,t) = \frac{2}{r}\frac{d^2}{dt^2} I^{ij}(t-r)}$$

We want $\bar{h}^{ij}_{TT}$, not $\bar{h}^{ij}$ in general gauge.

Introduce projection operator $P_{\ell m} \equiv \delta_{\ell m} - n_\ell n_m$ $\quad n_\ell \equiv x_\ell / r$

Projects out longitudinal piece.

Then $$\bar{h}^{TT}_{ij} = \underbrace{P_{i\ell} P_{jm} \bar{h}^{\ell m}}_{\text{Transverse}} - \underbrace{\tfrac{1}{2} P_{ij} P_{\ell m} \bar{h}^{\ell m}}_{\text{Traceless}}$$

(Check $P_{\ell j} P^{jm} = P_\ell{}^m$)

Then
$$\bar{h}^{TT}_{ij} = \frac{2}{r}\frac{d^2}{dt^2} I^{TT}_{ij}(t-r)$$

where $I^{TT}_{ij} \equiv P_{i\ell} P_{jm} I^{\ell m} - \frac{1}{2} P_{ij} (P_{\ell m} I^{\ell m})$

Finally, define

$$\mathcal{I}_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij} I$$

$$= \int T^{00}\left(x_i x_j - \frac{1}{3}\delta_{ij} r^2\right) d^3x$$

Note that $\mathcal{I}_{ij}^{TT} = I_{ij}^{TT}$

so $$\boxed{h^{TT}_{ij} = \frac{2}{r}\ddot{\mathcal{I}}_{ij}^{TT}(t-r)}$$

- Why $\mathcal{I}_{jk}$ instead of I_{jk}?

Consider Newtonian source, observer in near zone $r \ll \lambda$.

Newtonian potential is $\Phi = -\frac{1}{2}h_{00} = -\frac{1}{2}h^{00} = -\frac{1}{2}\left(\bar{h}^{00} + \frac{1}{2}\bar{h}\right) = -\frac{1}{4}\left(\bar{h}^{00} + \bar{h}^i{}_i\right)$

$$= -\int \frac{\left(T^{00} + t^{00} + T^i{}_i + t^i{}_i\right)_{ret} d^3x'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$= -\int \frac{T^{00}}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d^3x$$ bc $t^{00} + T^i{}_i + t^i{}_i \ll T^{00}$ for weak source

Note $$\frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} = \frac{1}{r} + \frac{x_j x'^j}{r^3} + \frac{1}{2}\frac{x_j x_k\left(3x'^j x'^k - \delta^{jk} r'^2\right)}{r^5} + \dots$$

(expand in powers of $\frac{1}{r}$)

So $\Phi = -\left(\frac{M}{r} + \frac{d_j x^j}{r^3} + \frac{3}{2}\frac{I_{ij} x^i x^j}{r^5} + \dots\right)$

where $M \equiv \int T^{00} d^3x$

$d_j \equiv \int T^{00} x_j \, d^3x$

So I_{ij} is measurable from near-zone gravitational field.

I_{ij} = <u>Reduced Quadrupole moment</u>

Shortwave approximation

Assume $g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu}$ "steady coordinates"

Background waves

where
- background is vacuum
- $A \equiv$ amplitude of waves $\ll 1$ so $|h_{\mu\nu}| \lesssim |g^{(B)}_{\mu\nu}| A$
- $R \equiv$ radius of curvature of background $\ll \lambda\!\!\!^{-}$ so $|g^{(B)}_{\mu\nu,\lambda}| \lesssim |g^{(B)}_{\mu\nu}| /$

  & $|h_{\mu\nu,\lambda}| \lesssim |h_{\mu\nu}| \lambda\!\!\!^{-}$

- Think of $g_{\mu\nu}$ + $g^{(B)}_{\mu\nu}$ as different metrics on same manifold.
- Raise/lower $h_{\mu\nu}$ using $g^{(B)}_{\mu\nu}$ (corrections are higher order)

Let ∇ = cov. deriv. wrt $g_{\mu\nu}$
$\nabla^{(B)}$ = cov. deriv. wrt $g^{(B)}_{\mu\nu}$

Note, we know that $\langle \tilde{\omega}^{\lambda}, \nabla_{\vec{e}_\beta} \vec{e}_\alpha \rangle = \Gamma^{\lambda}{}_{\alpha\beta}$ is not a tensor

But define $\tilde{\tilde{S}}$ such that $\tilde{\tilde{S}}(\tilde{u}, \vec{v}, \vec{w}) \equiv \langle \tilde{u}, \nabla_{\vec{v}} \vec{w} \rangle - \langle \tilde{u}, \nabla^{(B)}_{\vec{v}} \vec{w} \rangle$

Lemma: $\tilde{\tilde{S}}$ is a tensor.

PF. $$\tilde{\tilde{S}}(\tilde{u}, a\vec{v}, \vec{w}) = \langle \tilde{u}, \nabla_{\vec{w}}(a\vec{v}) \rangle - \langle \tilde{u}, \nabla^{(B)}_{\vec{w}}(a\vec{v}) \rangle$$
$$= a\tilde{\tilde{S}}(\tilde{u}, \vec{v}, \vec{w}) + \langle \tilde{u}, \vec{v} \rangle \nabla_{\vec{w}} a - \langle \tilde{u}, \vec{v} \rangle \nabla^{(B)}_{\vec{w}} a$$

cancel b/c a = scalar

$$= a\tilde{\tilde{S}}(\tilde{u}, \vec{v}, \vec{w})$$

linear ☐

$$S^{\alpha}{}_{\beta\gamma} = \tilde{\tilde{S}}(\tilde{\omega}^{\alpha}, \vec{e}_{\beta}, \vec{e}_{\gamma}) = \langle \tilde{\omega}^{\alpha}, \nabla_{\gamma} e_{\beta} \rangle - \langle \tilde{\omega}^{\alpha}, \nabla_{\gamma}^{(B)} e_{\beta} \rangle$$

$$= \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{(B)\alpha}{}_{\beta\gamma}$$

<u>components of a tensor</u>

In LLF of $g_{\mu\nu}^{(B)}$, $\Gamma^{\alpha}{}_{\beta\gamma}^{(B)} = 0$

$$S^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} (g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})$$

$$g_{\mu\beta,\gamma} = \underbrace{\cancel{g^{(B)}_{\mu\beta,\gamma}}}_{\to 0 \text{ in LLF of B}} + h_{\mu\beta,\gamma}$$

$$\underset{\text{in LLF}}{=} h_{\mu\beta|\gamma} \qquad \text{where } a|_{\beta} \text{ means } \nabla^{(B)}_{\vec{e}_{\beta}} a$$

So in LLF of $g_{\alpha\beta}(B)$,
$$\boxed{S^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} (h_{\mu\beta|\gamma} + h_{\mu\gamma|\beta} - h_{\beta\gamma|\mu})}$$

But This is a tensor eqn. True in all frames!

Lemma: $g^{\mu\nu} = g^{\mu\nu(B)} - h^{\mu\nu} + h^{\mu\alpha} h_{\alpha}{}^{\nu} - h^{\mu\alpha} h_{\alpha}{}^{\beta} h_{\beta}{}^{\nu} + \ldots$

PF $(A+B)^{-1} = A^{-1} - A^{-1} B (A+B)^{-1}$

$$= A^{-1} - A^{-1} B [A^{-1} + A^{-1} B [A^{-1} + A^{-1} B [\ldots$$

$$= A^{-1} - A^{-1} B A^{-1} + A^{-1} B A^{-1} B A^{-1} - \ldots$$

use $A = g_{\alpha\beta}^{(B)}$
$B = h_{\alpha\beta}$

Now, in LLF of (B), $R^{\alpha}{}_{\beta\gamma\delta} - R^{(B)\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\gamma\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\beta\gamma}$

$$- \Gamma^{(B)\alpha}{}_{\beta\delta,\gamma} + \Gamma^{(B)\alpha}{}_{\beta\gamma,\delta}$$

$$= S^{\alpha}{}_{\beta\delta,\gamma} - S^{\alpha}{}_{\beta\gamma,\delta} + S^{\alpha}{}_{\gamma\sigma} S^{\sigma}{}_{\beta\delta} - S^{\alpha}{}_{\delta\sigma} S^{\sigma}{}_{\beta\gamma}$$

$$= S^{\alpha}{}_{\beta\delta|\gamma} - S^{\alpha}{}_{\beta\gamma|\delta} + S^{\alpha}{}_{\gamma\sigma} S^{\sigma}{}_{\beta\delta} - S^{\alpha}{}_{\delta\sigma} S^{\sigma}{}_{\beta\gamma}$$

tensor eq. true in all frames

Also $R_{\beta\delta} - R^{(B)}_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} - R^{(B)\alpha}{}_{\beta\alpha\delta}$

$$= S^{\alpha}{}_{\beta\delta|\alpha} - S^{\alpha}{}_{\beta\alpha|\delta} + S^{\alpha}{}_{\alpha\sigma} S^{\sigma}{}_{\beta\delta} - S^{\alpha}{}_{\delta\sigma} S^{\sigma}{}_{\beta\alpha}$$

$$= R^{(1)}_{\beta\delta} + R^{(2)}_{\beta\delta} + \dots$$

linear in h   quadratic in h

$$S^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\left(g^{\alpha\mu(B)} - h^{\alpha\mu} + h^{\alpha\sigma}h_{\sigma}{}^{\mu} - \dots\right)\left(h_{\mu\beta|\gamma} + h_{\mu\gamma|\beta} - h_{\beta\gamma|\mu}\right)$$

Linear term: $R^{(1)}_{\beta\delta} = S^{\alpha}{}_{\beta\delta|\alpha} - S^{\alpha}{}_{\beta\alpha|\delta}$

$$= \frac{1}{2} g^{\alpha\mu(B)}\Big[\cancel{h_{\mu\beta|\delta\alpha}} + h_{\mu\delta|\beta\alpha} - h_{\beta\delta|\mu\alpha}$$

$$- \cancel{h_{\mu\beta|\alpha\delta}} - h_{\mu\alpha|\beta\delta} + h_{\beta\alpha|\mu\delta}\Big]$$

(commute to 1st order)

$$= \frac{1}{2}\left[-h_{|\beta\delta} - h_{\beta\delta|\alpha}{}^{\alpha} + h_{\alpha\beta|\delta}{}^{\alpha} + h_{\alpha\delta|\beta}{}^{\alpha}\right]$$

Quadratic term: you get

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[\frac{1}{2} h_{\alpha\beta|\mu} h^{\alpha\beta}{}_{|\nu} + h^{\alpha\beta}\left(h_{\alpha\beta|\mu\nu} + h_{\mu\nu|\alpha\beta} - h_{\alpha\mu|\nu\beta} - h_{\alpha\nu|\mu\beta}\right)$$

$$+ h_{\nu}{}^{\alpha|\beta}\left(h_{\alpha\mu|\beta} - h_{\beta\mu|\alpha}\right) - \left(h^{\alpha\beta}{}_{|\beta} - \frac{1}{2} h^{|\alpha}\right)\left(h_{\nu\alpha|\mu} + h_{\alpha\mu|\nu} - h_{\mu\nu|\alpha}\right)\Big]$$

Einstein's Eqs:

$$0 = R_{\mu\nu} = \underbrace{R^{(B)}_{\mu\nu}}_{A/\lambda^2} + \underbrace{R^{(1)}_{\mu\nu}(h)}_{A^2/\lambda^2} + \underbrace{R^{(2)}_{\mu\nu}(h)}_{A^3/\lambda^2} + \cdots$$

Now let $h_{\mu\nu} \Rightarrow \underbrace{h_{\mu\nu}}_{\text{linear in } A} + \underbrace{j_{\mu\nu}}_{A^2} + \underbrace{k_{\mu\nu}}_{A^3} + \cdots$

Solve to each order in A

$O(A)$: $\quad 0 = R^{(1)}_{\mu\nu}(h) \quad$ I $\quad$ Linearized theory recovered

$O(A^2)$: $\quad 0 = R^{(B)}_{\mu\nu} + R^{(1)}_{\mu\nu}(J) + R^{2}_{\mu\nu}(h) = 0$

split into smooth + wiggling part:

$$R^{(B)}_{\mu\nu} + \langle R^{(2)}_{\mu\nu}(h)\rangle = 0 \qquad \text{smooth} \qquad \text{II}$$

$$R^{(1)}_{\mu\nu}(J) + R^{(2)}_{\mu\nu}(h) - \langle R^{(2)}_{\mu\nu}(h)\rangle = 0 \qquad \text{wiggly} \qquad \text{III}$$

in Lorentz gauge:

$$\bar{h}_{\mu\nu|\alpha}{}^{\alpha} + 2R^{(B)}{}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} = 0$$

$$\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0$$

I: propagation of GWs

II: Energy in GWs curve the background

(Recall $A^{\alpha;\beta}{}_{\beta} - R^{\alpha}_{\beta}A^{\beta} = 0$ Maxwell)

$$G^{(B)}_{\mu\nu} = R^{(B)}_{\mu\nu} - \frac{1}{2}R^{(B)}g^{(B)}_{\mu\nu}$$

$$\equiv 8\pi T^{(GW)}_{\mu\nu}$$ Effective stress-energy

$$\text{So } T_{\mu\nu}^{(GW)} = -\frac{1}{8\pi}\left\{\langle R^{(2)}_{\mu\nu}(h)\rangle - \frac{1}{2}g_{\mu\nu}^{(B)}\langle R^{(2)}(h)\rangle\right\}$$

$$\Rightarrow \boxed{T_{\mu\nu}^{(GW)} = \frac{1}{32\pi}\langle h_{\alpha\beta|\mu}h^{\alpha\beta}{}_{|\nu}\rangle \quad (\text{in TT gauge})}$$

III Waves generate nonlinear corrections to themselves.

Recall energy in GW is

$$T_{\mu\nu}^{(GW)} = \frac{1}{32\pi} \left\langle h_{\alpha\beta|\mu} h^{\alpha\beta}{}_{|\nu} \right\rangle$$

Far from the source, $h_{ij}^{TT} = \frac{2}{r} \ddot{I}_{ij}^{TT}(t-r)$

$$\text{so} \quad T_{00}^{(GW)} = \frac{1}{32\pi} \left\langle h_{ij|0} h^{ij}{}_{|0} \right\rangle = \frac{1}{8\pi r^2} \left\langle \dddot{I}_{ij}^{TT} \dddot{I}^{ij\,TT} \right\rangle$$

$$\text{and} \quad T_{0r}^{(GW)} = -T_{00}^{(GW)} = -T_{rr}^{(GW)}$$

Total power crossing sphere of radius r at time t is

$$L_{GW}(t,r) = \int T^{0r\,(GW)} r^2 \, d\Omega$$

$$= \frac{1}{5} \left\langle \dddot{I}_{ij}(t-r) \, \dddot{I}^{ij}(t-r) \right\rangle$$

Similarly, can show total ang momentum flux crossing sphere

$$\frac{dJ_i}{dt} = +\frac{2}{5} \varepsilon_{ijk} \left\langle \ddot{I}^{j\ell} \dddot{I}_{\ell}{}^{k} \right\rangle$$