Conservation Laws in linearized Theory

Define $H^{\mu\alpha\nu\beta} \equiv -\left(\bar{h}^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\nu}\bar{h}^{\alpha\beta} - \bar{h}^{\alpha\nu}\eta^{\mu\beta} - \bar{h}^{\mu\beta}\eta^{\alpha\nu}\right)$

Same symmetries as Riemann: $H^{\mu\alpha\nu\beta} = H^{\nu\beta\mu\alpha}$

$$= H^{[\mu\alpha][\nu\beta]}$$

$$H^{\mu[\alpha\nu\beta]} = 0$$

$H^{\mu\alpha\nu\beta}$ is <u>not</u> a tensor

(it is a tensor in linearized Theory under Lorentz xforms, but is not gauge-invariant)

Then $\boxed{H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 2G^{\mu\nu} = 16\pi T^{\mu\nu}}$ (linearized)

And $H^{\mu\alpha\nu\beta}{}_{,\alpha\beta\nu} = 0$ (antisymmetry)

$$= 16\pi T^{\mu\nu}{}_{,\nu}$$

so $T^{\mu\nu}{}_{,\nu} = 0$

So look at $P^{\mu} = \int T^{\mu 0}\, d^3x$

H antisym

$$= \frac{1}{16\pi}\int H^{\mu\alpha 0\beta}{}_{,\alpha\beta}\, d^3x = \frac{1}{16\pi}\int H^{\mu\alpha 0 j}{}_{,\alpha j}\, d^3x$$

$$= \frac{1}{16\pi}\oint H^{\mu\alpha 0 j}{}_{,\alpha}\, d^2S_j \quad \text{(Gauss)}$$

Similarly,

$$J^{\mu\nu} = \int (x^\mu T^{\nu 0} - x^\nu T^{\mu 0})\, d^3x$$

$$= \frac{1}{16\pi} \int 2x^{[\mu} H^{\nu]\alpha 0\beta}{}_{,\alpha\beta}\, d^3x$$

$$= \frac{1}{8\pi} \oint \left(x^\mu H^{\nu\alpha 0 i}{}_{,\alpha} + H^{0[\nu\mu]i}\right) d^2S_i$$

Integrands not gauge-inv.
Integrals are

Then $M = (-P^\mu P_\mu)^{1/2}$, $S^\alpha = -\frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} U_\delta$

$U^\alpha = P^\alpha / M$,

Now Full GR

Define $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$

Define $H^{\mu\alpha\nu\beta}$ as before

not perturbative
h can be large

Then $P^\mu = \frac{1}{16\pi} \oint H^{\mu\alpha 0 j}{}_{,\alpha}\, d^2S_j$ as before.

$J^{\mu\nu} = \frac{1}{8\pi} \oint \left(x^\mu H^{\nu\alpha 0 i}{}_{,\alpha} + H^{0[\mu\nu]i}\right) d^2S_i$ as before.

Integral is in flat region

But we can still then say $P^\mu = \frac{1}{16\pi} \int H^{\mu\alpha 0\beta}{}_{,\alpha\beta}\, d^3x$

$$J^{\mu\nu} = \frac{1}{8\pi} \int x^{[\mu} H^{\nu]\alpha 0\beta}{}_{,\alpha\beta}\, d^3x$$

(Gauss)

Let $T^{\mu\nu}_{\rm eff} \equiv \frac{1}{16\pi} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}$

$$\Rightarrow \quad P^{\mu} = \int T^{\mu 0}{}_{\text{eff}}\, d^3x$$

$$J^{\mu\nu} = 2\int x^{[\mu} T^{\nu]0}{}_{\text{eff}}\, d^3x$$

define $\quad t^{\mu\nu} \equiv \frac{1}{16\pi}\underbrace{\left[H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} - 2G^{\mu\nu}\right]}_{\text{Not zero in full GR}}$

Then $\quad H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi t^{\mu\nu} + 2G^{\mu\nu}$

$$= 16\pi\left(t^{\mu\nu} + T^{\mu\nu}\right)$$

So $\quad T^{\mu\nu}{}_{\text{eff}} = T^{\mu\nu} + t^{\mu\nu}$

$t^{\mu\nu}$ = stress-energy pseudotensor.
describes gravitational stress-energy

<u>Not a tensor</u>

Notice, commas (ordinary derivs) in above eqs.

Also note $T^{\mu\nu}{}_{\text{eff},\nu} = 0$

Nonuniqueness

Notice that *any* $H^{\mu\alpha\nu\beta}$ that reduces to

$$-\left(\bar{h}^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\nu}\bar{h}^{\alpha\beta}-\bar{h}^{\alpha\nu}\eta^{\mu\beta}-\bar{h}^{\mu\beta}\eta^{\alpha\nu}\right)$$

far from source will give the same $P^{\alpha}, J^{\alpha\beta}, S_{\alpha}, M$

For example, can choose

$$H^{\mu\alpha\nu\beta}_{LL} = (-g)\left[g^{\mu\nu}g^{\alpha\beta}-g^{\alpha\nu}g^{\mu\beta}\right]$$

$$\Rightarrow \quad H^{\mu\alpha\nu\beta}_{LL\ \ ,\alpha\beta} = 16\pi\,(-g)\,(T^{\mu\nu}+t^{\mu\nu}_{LL})$$

Landau-Lifshitz Pseudotensor

Same as $H^{\mu\alpha\nu\beta}$ far from source.

$t^{\mu\nu}_{LL}$ convenient b/c quadratic in 1st derivs of metric

---

So $\quad T^{\mu\nu}_{LL\,\mathrm{eff}} \equiv (-g)(T^{\mu\nu}+t^{\mu\nu}_{LL})$ equals $T^{\mu\nu}_{(\mathrm{eff})}$ far from source.

What about volume integrals?

$$P^{\mu} = \int T^{\mu 0}_{LL\,\mathrm{eff}}\, d^3x = \int T^{\mu 0}_{\mathrm{eff}}\, d^3x$$

but locally inside source, $\boxed{T^{\mu 0}_{LL\,\mathrm{eff}} \neq T^{\mu 0}_{\mathrm{eff}}}$

Energy-momentum of grav. field not localizable.

In asymptotic rest frame, $\vec{S}$ + $\vec{P}$ are 4-vectors $\quad \vec{S}\cdot\vec{P} = 0$

What about multiple sources?

$$P^\mu = \int T^{\mu 0}_{\text{eff}}\, d^3x$$

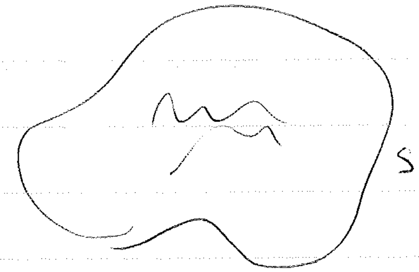
$$= \sum_{\text{sources}} \int_{\text{source}} T^{\mu 0}_{\text{eff}}\, d^3x + \int_{\text{between}} T^{\mu 0}_{\text{eff}}\, d^3x$$

$$= \sum_{\text{sources}} P^\mu_{\text{source}} + \int_{\text{between}} T^{\mu 0}_{\text{eff}}\, d^3x$$

Can measure each source separately, + add flat region

Need flat region between sources

Conservation Laws



$$P^{\mu} = \int T^{\mu 0}_{\text{eff}}\, d^3x \qquad \text{inside } S$$

$$\frac{dP^{\mu}}{dt} = \int T^{\mu 0}_{\text{eff},0}\, d^3x = -\int T^{\mu i}_{\text{eff},i}\, d^3x$$

$$= -\oint T^{\mu i}_{\text{eff}}\, d^2S_i$$

Flux integral ~~[illegible]~~ in asym. flat region

Similarly

$$\cancel{x^{[\mu}{}_{,\alpha} T^{\nu]\alpha}_{\text{eff}}}^{\,0} + \cancel{x^{[\mu} T^{\nu]\alpha}_{\text{eff},\alpha}}^{\,0}$$

$$J^{\mu\nu} = \int 2 x^{[\mu} T^{\nu]0}_{\text{eff}}\, d^3x$$

$$\frac{dJ^{\mu\nu}}{dt} = \int 2\left(x^{[\mu} T^{\nu]0}_{\text{eff}}\right)_{,0} d^3x = -2\int \left(x^{[\mu} T^{\nu]i}_{\text{eff}}\right)_{,i} d^3x \quad + 2\int \left(x^{[\mu} T^{\nu]\alpha}_{\text{eff}}\right)_{,\alpha} d^3x$$

$$= -2\oint x^{[\mu} T^{\nu]i}_{\text{eff}}\, d^2S_i$$

More on Gravitational Waves

Recall: $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ ← SMALL

$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

Linearized theory

$h \equiv h^{\alpha}{}_{\alpha}$

Then

Lorentz gauge $\Rightarrow$ $\boxed{\bar{h}^{\mu\alpha}{}_{,\alpha} = 0}$

Then Einstein's Eqs $\Rightarrow$ $\boxed{\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -16\pi T_{\mu\nu}}$ Linearized Einstein Eqs

Still gauge freedom: $X^{\mu}_{new} = X^{\mu}_{old} + \xi^{\mu}$ ← SMALL

$$\Rightarrow \bar{h}_{\mu\nu}^{new} = \bar{h}_{\mu\nu}^{old} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi^{\alpha}{}_{,\alpha}$$

Can choose Transverse Traceless (TT) gauge:

$$\boxed{\bar{h}_{\mu 0} = 0 \quad \text{and} \quad \bar{h}_{\mu}{}^{\mu} = 0}$$

Then if $\bar{h}_{\alpha\beta} = \underbrace{A_{\alpha\beta}}_{const} e^{i k_{\mu} x^{\mu}}$

Note: $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$ in TT gauge

Then $A_{\mu\nu} k^{\nu} = 0$ Lorentz

$A^{\alpha}{}_{\alpha} = 0$

$A_{\alpha 0} = 0$

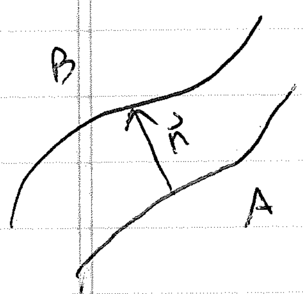
if $k_i = k\hat{z}$

then $ds^2 = -dt^2 + (1 + A_+ e^{i k_\alpha x^\alpha}) dx^2 + (1 - A_+ e^{i k_\alpha x^\alpha}) dy^2 + 2 A_\times e^{i k_\alpha x^\alpha} dx\,dy + dz^2$

Effect of GW on test particles

$$R_{j0k0} = -\frac{1}{2} h^{TT}_{jk,00} \qquad \text{others zero}$$

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0$$



choose orthonormal frame comoving w/ particle A, orthonormal FW transported tetrad

(LLF of A)

$$X^{\hat{j}} = 0$$

$$X^{\hat{0}} = \tau$$

$\vec{n} \cdot \vec{u}_A = 0$ so $n^{\hat{0}} \equiv 0$

$$\Rightarrow \quad \frac{D^2 n^{\hat{j}}}{d\tau^2} = -R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} n^{\hat{k}} = -R_{\hat{j}\hat{0}\hat{k}\hat{0}} n^{\hat{k}}$$

at A, Γ and $\frac{d\Gamma}{d\tau} = 0$

$$\Rightarrow \quad \frac{d^2 x_B^{\hat{j}}}{d\tau^2} = -R_{\hat{j}\hat{0}\hat{k}\hat{0}} n^{\hat{k}}$$

Choose TT coord system coincident w/ particle A $\qquad t = \tau$ to 1st order

$$\Rightarrow \quad \frac{d^2 x_B^{\hat{j}}}{dt^2} = +\frac{1}{2} h^{TT}_{jk,00} X_B{}^k$$

Suppose $X_B^{\hat{j}} = x_{B(0)}^{\hat{j}}$ at $t \to -\infty$ & particles at rest before wave arrives

$$\Rightarrow \quad X_B^{\hat{j}}(t) = X_{B(0)}{}^{\hat{k}} \left[\delta_{\hat{j}\hat{k}} + \frac{1}{2} h_{jk}^{TT} \right]$$

Polarization of GWs

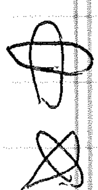
In TT Gauge, plane wave propagating in z direction:

$$h_{0\mu} = 0 \ , \ h_{iz} = 0$$

$$h_{xx} = -h_{yy} = \mathrm{Re}\left(A_+ e^{-i\omega(t-z)}\right)$$

$$h_{xy} = \mathrm{Re}\left(A_\times e^{-i\omega(t-z)}\right)$$

2 polarizations.



let $\tilde{\tilde{e}}_{(+)} \equiv \vec{e}_x \otimes \vec{e}_x - \vec{e}_y \otimes \vec{e}_y$

$\tilde{\tilde{e}}_{(\times)} \equiv \vec{e}_x \otimes \vec{e}_y + \vec{e}_y \otimes \vec{e}_x$

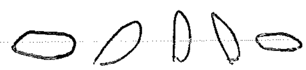
Polarization tensors

then $$h_{\mu\nu} = \mathrm{Re}\left(\left(A_+ e^{(+)}_{\mu\nu} + A_\times e^{(\times)}_{\mu\nu}\right) e^{-i\omega(t-z)}\right)$$

also circular polarization:

$$\tilde{\tilde{e}}_{(R)} = \frac{1}{\sqrt{2}}\left(\tilde{\tilde{e}}^{(+)} + i\,\tilde{\tilde{e}}^{(\times)}\right)$$

$$\tilde{\tilde{e}}_{(L)} = \frac{1}{\sqrt{2}}\left(\tilde{\tilde{e}}^{(+)} - i\,\tilde{\tilde{e}}^{(\times)}\right)$$



GWs in Full GR

Nonlinear effects:

- GWs have energy, so $T^{\mu\nu}_{eff(GW)}$ curves the background spacetime.
- Radiating source dissipates energy.
- Waves can be redshifted, refracted, scattered off spacetime curvature.
- Wave can affect itself thru curvature