Dynamical black holes

So far, we have dealt with <u>stationary</u> BHs.
But BHs can absorb mass, charge, ang momentum.

- In general, dynamical BHs must be treated numerically (later in course)
- There are some global Theorems (e.g. Laws of BH Thermo)

One simple example: Vaidya spacetime.

- Spherically symmetric
- "null dust" matter source

Vaidya

First, write down Schwarzschild in IEF coords:

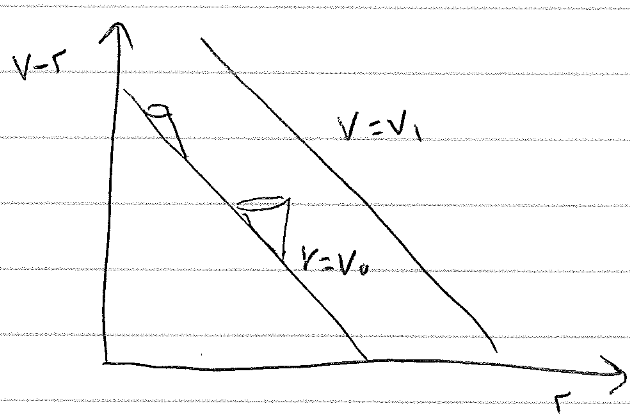
$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2 \qquad \left(v = t - r^* \quad dr^* = \frac{dr}{1-\frac{2M}{r}} \right)$$

Now allow $M \to M(v)$

$$\boxed{ds^2 = -\left(1-\frac{2M(v)}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2}$$

Ingoing Vaidya metric

v is ingoing null coord. Mass changes with v



Matter falling into BH along ingoing null rays.

"Poor man's way to solve Einstein Eqs"

From Vaidya, compute $\Gamma^{\alpha}{}_{\beta\gamma}$, compute $R_{\alpha\beta\gamma\delta}$, compute $G_{\alpha\beta}$

you get

$$\Rightarrow G_{vv} = \frac{1}{2r^2}\frac{dM(v)}{dv}$$

$$\text{So} \quad T_{vv} = \frac{dM}{dv}\frac{1}{4\pi r^2} \quad \text{or} \quad \boxed{T_{\alpha\beta} = \frac{dM}{dv}\frac{1}{4\pi r^2}\, l_\alpha l_\beta}$$

where $l_\alpha \equiv -\partial_\alpha V$

l_α is ingoing null geodesic.

$T_{\alpha\beta}$ describing "Null dust" — good appx to radiation in high-freq, geometric optics limit.

Let's look for <u>apparent</u> horizons:

Recall (spherical symmetry)

$\frac{d(\text{Area})}{d\lambda}$ of outgoing spherical pulse of light can be >0 or <0

$\frac{d(\text{Area})}{d\lambda} < 0$ means surface is <u>Trapped</u>

AH is outermost marginally trapped surface

(Better defn of AH given later in class)

$$ds^2 = 0 = -f\,dv^2 + 2\,dr\,dv \qquad f = 1 - \frac{2M(v)}{r}$$

$$\text{Area} = 4\pi r^2 \quad \text{so} \quad \frac{d\,\text{Area}}{d\lambda} \propto \frac{dr}{d\lambda} = \frac{dr}{dv}\frac{dv}{d\lambda}$$

$$\text{From } ds^2 = 0, \quad -f + 2\frac{dr}{dv} = 0 \qquad \frac{dr}{dv} = \frac{f}{2} = \frac{1}{2} - \frac{M(v)}{r}$$

$$\text{So AH when } \frac{dr}{dv} = 0 \quad \text{or} \quad \boxed{2M(v) = r}_{\,AH}$$

Now suppose $M(v) = \begin{cases} M_1 & v < v_1 \\ M_{12}(v) & v_1 < v < v_2 \\ M_2 & v > v_2 \end{cases}$

Assume $M_2 > M_1$, $\frac{dM_{12}}{dv} > 0$

So for $v < v_1$ and $v > v_2$ we have Schwarzschild.

AH is at $r = 2M_1$ $(v < v_1)$ null

$r = 2M_2$ $(v > v_2)$ null

but what about $v_1 < v < v_2$?

Let $\Phi = r - 2M_{12}(v)$ so $\Phi = 0$ is a surface that describes the AH

Then $\Phi_{,r} = 1$ $\Phi_{,v} = -2\frac{dM_{12}}{dv}$

So normal to surface is $\Phi_{;\alpha}$ (or $\tilde{\partial}\Phi$)

Magnitude is $g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} = g^{vv}\Phi_{,v}\Phi_{,v} + 2g^{vr}\Phi_{,v}\Phi_{,r} + g^{rr}\Phi_{,r}\Phi_{,r}$

$g^{rr} = f$

$g^{rv} = 1$

$g^{vv} = 0$

so $g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} = -4\frac{dM_{12}}{dv} + f$

but $f = 0$ on AH

so $g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} = -4\frac{dM_{12}}{dv}$ < 0 if $\frac{dM_{12}}{dv} > 0$

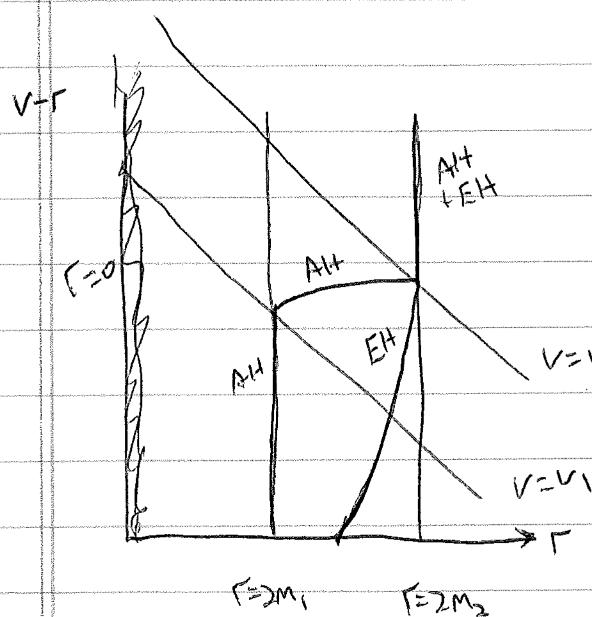
$\Rightarrow$ Normal to AH is timelike. AH is spacelike

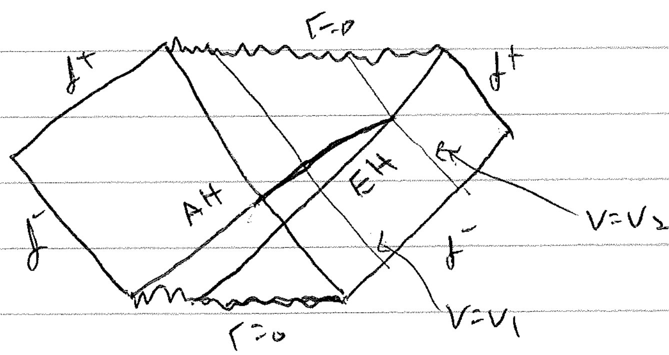
What about <u>event</u> horizon?

For $v > v_2$, Schwarzschild $\Rightarrow$ EH at $r = 2M_2$

EH generated by photon that stays on horizon

So for $v < v_2$ EH is outgoing null geodesic that smoothly matches $r = 2M_2$ at $v \geq v_2$



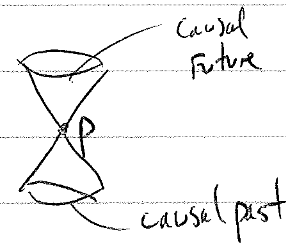


Notice - EH has $\frac{dr}{dv} > 0$ for $v < v_2$

- AH inside EH

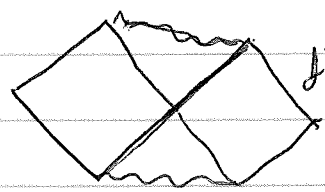
More general definitions:

Given pt. p, <u>causal future</u> of p is set of pts that can be reached from p, by future null or timelike curves



<u>causal past</u> of p is set of pts that can ~~reach~~ reach p, by future-directed null or timelike curves.

A Black Hole is the set of pts <u>not</u> in the causal past of $\mathscr{I}^+$



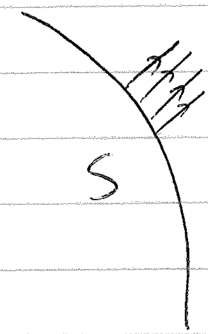
<u>Event Horizon</u> is bdry of black hole.

EH is generated by null geodesics with no future endpoints.

- generator may or may not leave the EH when followed into past.
- generator cannot leave the EH when followed into the future
- two generators never intersect, except possibly where they enter the EH
- Except for entry points, each pt on EH has only 1 generator passing thru it.

Apparent horizon

Given 2-surface S, consider set ("congruence") of outgoing null geodesics normal to S.



At every pt near S, there is one of these geodesics. They have tangent $\vec{k}$ (affinely parameterized)

Let expansion $= \theta \equiv k^{\alpha}{}_{;\alpha}$

Outer-Trapped Surface : Closed 2-surface S s.t. $\theta < 0$ everywhere on S

Marginally Outer-Trapped Surface: " " $\theta = 0$ " "

An AH is the outermost MOTS.

THM: AH is inside or on EH (if AH exists, + cosmic censorship holds)

THM: if AH exists, there is a singularity inside.

Mass + Spin for fully relativistic, <u>asym. flat</u> system.

Assume: - Linearized Theory far from source
- Vacuum far from source
- Choose center of coords so total linear momentum $P^i = 0$

$$g_{\mu\nu} = \underbrace{h_{\mu\nu}}_{\text{small}} + \eta_{\mu\nu}$$
$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r} + \frac{2M^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) dt^2 - \left[4 \varepsilon_{ik\ell} S^k \frac{x^\ell}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right)\right] dt\, dx^i$$

$$+ \left[\left(1 + \frac{2M}{r} + \frac{3M^2}{2r^2}\right)\delta_{ij} + \text{GW terms}\right] dx^i dx^j$$

where $M, S^k = \text{const}$ GW terms go like $\frac{1}{r}$

For weak source, $M = \int T^{00} d^3x$

$$P^i = \int T^{0i} d^3x = 0$$

$$S_i = \int \varepsilon_{ijk} x^j T^{k0} d^3x$$

For strong source, M, S, P are just metric coefficients

<u>Above integrals fail</u>. cannot do integrals over source <u>self</u> <u>gravity</u>

Physically, M & S measured by orbits, frame dragging / precession