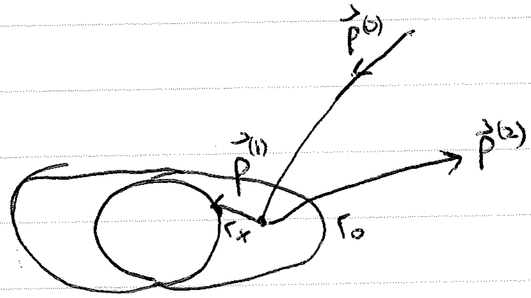
Energy extraction + the Penrose process

$$E = -\vec{p}\cdot\vec{\frac{\partial}{\partial t}} \qquad \text{conserved energy along geodesic}$$

Inside ergosphere, $\vec{\frac{\partial}{\partial t}}$ is spacelike, $\left(\text{i.e. } g_{tt} > 0 \text{ inside } r = r_0 = M + \sqrt{M^2 - a^2\cos^2\theta} \right)$

so it is possible to have $E < 0$.

An $E<0$ particle cannot escape to ∞, but consider:



- Particle $\vec{p}^{(0)}$ comes from ∞ into Ergosphere
- $\vec{p}^{(0)}$ breaks into 2 particles

$$\vec{p}^{(0)} = \vec{p}^{(1)} + \vec{p}^{(2)} \qquad \underline{\text{mom. conservation}}$$

assume $\vec{p}^{(1)}$ has negative energy

Then $\vec{p}^{(2)} = \vec{p}^{(0)} - \vec{p}^{(1)}$

$$\vec{\frac{\partial}{\partial t}}\cdot\vec{p}^{(2)} = \vec{\frac{\partial}{\partial t}}\cdot\vec{p}^{(0)} - \vec{\frac{\partial}{\partial t}}\cdot\vec{p}^{(1)}$$

$$-E^{(2)} = -E^{(0)} + E^{(1)}$$

Negative so $E^{(2)} > E^{(0)}$

<u>outgoing particle $\vec{p}^{(2)}$ has more energy than ingoing particle $\vec{p}^{(0)}$!</u>

"Penrose Process"

When can $E<0$?

From geodesic eqn,

$$\Sigma^2\left(p^{r^2}+\Delta p^{\theta^2}\right) = \left[(r^2+a^2)E-aL\right]^2 - \mu^2 r^2 \Delta$$

$$- \frac{\Delta}{\sin^2\theta}\left[L-aE\sin^2\theta\right]^2 - \mu^2 a^2 \Delta \cos^2\theta$$

$$\Rightarrow \quad \alpha E^2 - 2\beta E + \gamma = 0$$

where $\alpha = (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$ (always >0 outside r_+)

$\beta = 2MraL$

$\gamma = L^2\left(a^2 - \frac{\Delta}{\sin^2\theta}\right) - \mu^2\Delta\Sigma - \Sigma^2\left((p^r)^2+\Delta(p^\theta)^2\right)$

So $E = \dfrac{\beta + \sqrt{\beta^2-\alpha\gamma}}{\alpha}$ (negative sqrt corresponds to past-pointing 4-momentum)

Notice: $E<0$ only when $\beta<0$ and $\gamma>0$

$\Rightarrow$ $E<0$ means $\underline{L<0}$

Also for fixed L, r, a E is smaller if γ is larger. (even for $E>0$)

$\Rightarrow$ Minimum E occurs when $p^r \to 0$, $p^\theta \to 0$, $\mu \to 0$

Minimum E at horizon r_+ is then $E_{min} = \dfrac{La}{r_+^2+a^2}$

(plugging in $\Delta \to 0$ $r \to r_+$)

$\Omega_H = \Omega$ of stationary observer at horizon

Recall ang velocity of horizon is $\Omega_H = \dfrac{a}{2Mr_+} = \dfrac{a}{r_+^2+a^2}$

So $E_{min} = L\Omega_H$ at horizon

or $E \geq L\Omega_H$ at horizon

can be negative — must then be more negative

Where does energy come from? (in Penrose process)

Now imagine object at ∞ with $E + L$,
dropped into BH.

By Energy / ang-mom conservation:

Mass of BH

Ang mom of BH

$$\delta M = +E$$
$$\delta J = +L$$

BH gains spin / energy of particle dropped into it.
(Neglect GWs for now)

Can generalize: $\delta M = \sum E_{\text{All particles crossing horizon}}$
$\delta J = \sum L$ " "

From earlier relation $E \geq L\Omega_H$

$$\Rightarrow \boxed{\delta M \geq \Omega_H \delta J}$$

Now consider surface area of BH horizon

$$A = \int \sqrt{{}^{(2)}g}\, d\theta\, d\phi$$

$${}^{(2)}ds^2 = \Sigma\, d\theta^2 + \left(r^2+a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

For $r \to r_+$ $\quad 2Mr_+ \to r_+^2+a^2$

$$\text{so } {}^{(2)}ds^2 = \Sigma\, d\theta^2 + (r_+^2+a^2)\left[\frac{\overbrace{\Sigma + a^2\sin^2\theta}^{r_+^2+a^2}}{\Sigma}\right]\sin^2\theta\, d\phi^2$$

$$= \Sigma\, d\theta^2 + (r_+^2+a^2)^2 \frac{\sin^2\theta}{\Sigma}\, d\phi^2$$

$$\text{so } \sqrt{{}^{(2)}g} = \sin\theta\,(r_+^2+a^2)$$

$$\Rightarrow \boxed{A = 4\pi(r_+^2+a^2)}$$

$$= 8\pi M r_+$$

$(J = aM)$

$$= 8\pi\left(M^2 + M\sqrt{M^2-a^2}\right) = 8\pi\left[M^2 + \sqrt{M^4-J^2}\right]$$

$$\text{Let } M_{irr}^2 \equiv \frac{A}{16\pi} = \frac{1}{2}\left(M^2+\sqrt{M^4-J^2}\right)$$ "irreducible mass"

$$\text{Then } 2M_{irr}\,\delta M_{irr} = \frac{1}{2}\left(2M\delta M + \frac{2M^3\delta M - J\delta J}{\sqrt{M^4-J^2}}\right)$$

$$\Rightarrow \delta M_{irr} = \frac{aM}{4M_{irr}\sqrt{M^4-J^2}}\left[\frac{\delta M}{\Omega_H} - \delta J\right]$$

always > 0

bc $\delta M \geq \delta J\, \Omega_H$

This is 2nd law of BH dynamics

$$\delta A \geq 0$$

Proven by Hawking for all BHs
1971

Surface Gravity:

Consider $\vec{\xi} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$ KV on horizon

Define "surface gravity" $\kappa \equiv \sqrt{-\frac{1}{2} \xi^{\alpha;\beta} \xi_{\alpha;\beta}}$

Equivalently, $\xi^\beta \xi^\alpha{}_{;\beta} = \kappa \xi^\alpha$

and $\left(-\xi^\beta \xi_\beta\right)_{;\alpha} = 2\kappa \xi_\alpha$

κ is force required to hold particle at horizon, hanging from string at ∞.

let's evaluate κ for Kerr:

$$\xi^\alpha \xi_\alpha = \frac{(a^2+r^2)^2 - a^2 \Delta \sin^2\theta}{\Sigma} \sin^2\theta \left[\Omega_H + \frac{g_{t\phi}}{g_{\phi\phi}}\right]^2 - \frac{\Sigma \Delta}{(a^2+r^2)^2 - a^2 \Delta \sin^2\theta}$$

because $\Omega_H \to \frac{g_{t\phi}}{g_{\phi\phi}}$ on horizon and $\Delta \to 0$ on horizon

$$\left(\xi^\alpha \xi_\alpha\right)_{,\beta} = \frac{-\Sigma \Delta_{,\beta}}{(a^2+r^2)^2 - a^2 \Delta \sin^2\theta}$$

So $\left(-\xi^\alpha \xi_\alpha\right)_{;r} = \dfrac{+\Sigma}{(a^2+r_+^2)^2} 2(r_+ - M)$ other comps zero

What is ξ_α ?

$$\tilde{\xi} = \underbrace{\left(g_{tt} + g_{t\phi}\Omega_H\right)}_{\to 0 \text{ at } r_+} dt + \underbrace{\left(g_{t\phi} + \Omega_H g_{\phi\phi}\right)}_{\to 0 \text{ at } r_+} d\phi$$

but t, ϕ singular at horizon

So in these coords $\tilde{\xi} \to 0$ at horizon?

No, use Kerr coords $d\tilde{v} = dt + \dfrac{r^2+a^2}{\Delta} dr$

$$d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr$$

$$\Rightarrow \tilde{\xi} = d\tilde{v}\left[g_{tt} + g_{t\phi}\Omega_H\right] + d\tilde{\phi}\left[g_{t\phi} + g_{\phi\phi}\Omega_H\right]$$

$$- \frac{dr}{\Delta}\left[(r^2+a^2)(g_{tt} + g_{t\phi}\Omega_H) + a(g_{t\phi} + \Omega_H g_{\phi\phi})\right]$$

$$= d\tilde{v}\left[g_{tt} + g_{t\phi}\Omega_H\right] + d\tilde{\phi}\left[g_{t\phi} + g_{\phi\phi}\Omega_H\right]$$

$$+ \; dr\left[1 - a\Omega_H \sin^2\theta\right]$$

on $r = r_+$ $\quad \tilde{\xi} = dr\left[1 - a\Omega_H \sin^2\theta\right]$

$$\Omega_H = \frac{a}{r_+^2 + a^2}$$

So $\dfrac{\Sigma}{(a^2+r_+^2)^2} 2(r_+ - M) = 2K\left(1 - a\Omega_H \sin^2\theta\right)$

$$\boxed{K = \frac{r_+ - M}{r_+^2 + a^2}}$$

Look again at variations:

$$A = 4\pi(r_+^2 + a^2) \quad \Rightarrow \quad \frac{\delta A}{8\pi} = r_+ \delta r_+ + a\,\delta a$$

also $\quad r_+^2 + a^2 - 2Mr_+ = 0 \quad \Rightarrow \quad (r_+ - M)\delta r_+ + a\,\delta a - r_+ \delta M = 0$

(Eqn for r_+)

also $\quad J = aM \quad \Rightarrow \quad \delta J = a\,\delta M + M\,\delta a$

Eliminate $\delta a, \delta r_+$

$$\Rightarrow \boxed{\frac{\kappa}{8\pi}\,\delta A = \delta M - \Omega_H\,\delta J}$$

First law of BH dynamics

Turns out True for all BHs

Barden, Carter, Hawking 1973

Thermo	$\Leftrightarrow$	BH
S	$\Leftrightarrow$	$A/4$
T	$\Leftrightarrow$	$\kappa/2\pi$
E	$\Leftrightarrow$	M

(0th Law:
κ const on horizon)