Last time:

Given Lagrangian $L(x^\alpha, \dot{x}^\alpha)$ define $P_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha}$

Note $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}$
(E-L)

Then $H(x^\alpha, P_\alpha) = P_\alpha \dot{x}^\alpha - L$ Hamiltonian

$\Rightarrow$ Hamilton's Eqs

$$\boxed{\begin{aligned} \frac{dx^\alpha}{d\lambda} &= \frac{\partial H}{\partial P_\alpha} \\ \frac{dP_\alpha}{d\lambda} &= -\frac{\partial H}{\partial x^\alpha} \end{aligned}}$$

(For geodesics $H = \frac{1}{2} g^{\alpha\beta} P_\alpha P_\beta$
$L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$)

$\Rightarrow$ Hamilton-Jacobi

Canonical Xformation

Given - old coords/momenta x^α, P_α

- old Hamiltonian $H(x^\alpha, P_\alpha, \lambda)$

New momenta

- almost arbitrary "generating function" $F(x^\alpha, \bar{P}_\alpha, \lambda)$ $\left(\frac{\partial^2 F}{\partial x^\alpha \partial \bar{P}_\alpha} \neq 0 \right)$

Thm $$\bar{H}(\bar{x}^\alpha, \bar{P}_\alpha, \lambda) = H(x^\alpha, P_\alpha, \lambda) + \frac{\partial F}{\partial \lambda}$$

where $\bar{x}^\alpha \equiv \frac{\partial F}{\partial \bar{P}_\alpha}$ and also $P_\alpha = \frac{\partial F}{\partial x^\alpha}$

New Hamiltonian, new coords & momenta, same physics

Now suppose you can find an F s.t. $\bar{H} \equiv 0$

then $\frac{\partial \bar{H}}{\partial \bar{x}^\alpha} = 0 = \frac{d}{d\lambda} \bar{P}_\alpha$ $\bar{P}_\alpha = \text{const}$

$\frac{\partial \bar{H}}{\partial \bar{P}_\alpha} = 0 = \frac{d}{d\lambda} \bar{x}^\alpha$ $\bar{x}^\alpha = \text{const}$

Then $$0 = \frac{\partial F}{\partial \lambda} + H(x^\alpha, P_\alpha, \lambda)$$

$$= \frac{\partial F}{\partial \lambda} + H\left(x^\alpha, \frac{\partial F}{\partial x^\alpha}, \lambda\right)$$

let's rename $F(x^\alpha, \bar{P}_\alpha, \lambda) = S(x^\alpha, \bar{P}_\alpha, \lambda)$

$$\Rightarrow \boxed{\frac{\partial S}{\partial \lambda} + H\left(x^\alpha, \frac{\partial S}{\partial x^\alpha}, \lambda\right) = 0}$$ Hamilton Jacobi

and $$\frac{dS}{d\lambda} = \frac{\partial S}{\partial \lambda} + \frac{\partial S}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \lambda}$$

$$= -H + P_\alpha \frac{\partial x^\alpha}{\partial \lambda}$$

$$= L$$

so $\boxed{S = \int L \, d\lambda}$

And notice $\bar{x}^\alpha = \frac{\partial S}{\partial \bar{P}_\alpha}$ are constant

Jacobi Thm

So: Solve $\frac{\partial S}{\partial \lambda} + H(x^\alpha, \frac{\partial S}{\partial x^\alpha}, \lambda) = 0$ For $S(x^\alpha, \lambda)$

PDE.

$\Rightarrow$ Soln will depend on N constants (integration constants), call them $\bar{P}_\alpha$

So $S = S(x^\alpha, \lambda, \bar{P}_\alpha)$

Then (Jacobi Thm) $\frac{\partial S}{\partial \bar{P}_\alpha} \equiv \bar{x}^\alpha$ are also constants

Also, if $\frac{\partial H}{\partial x^1} = 0$ for some coord x^1

Then $\frac{dP_1}{d\lambda} = \frac{-\partial H}{\partial x_1} = 0$ $P_1 = \text{const}$

but $P_1 = \frac{\partial S}{\partial x^1}$

So $\boxed{S = P_1 x^1 + S(x^2, x^3, \ldots)}$

Also, if H indep of λ, then can write $S = -Et + \tilde{S}(x^\alpha)$

$H = \text{const},$

So $\frac{\partial S}{\partial t} = -H = -E$

E is one of the nontrivial constants

So $\frac{\partial S}{\partial E} = \text{const} = -t + \frac{\partial \tilde{S}}{\partial E}$

Example: 1d harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad m, \omega = \text{const}$$

Here $\lambda = t$

So H.J. $\Rightarrow$ $$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x = 0$$

Let's write $$S(x,t) = -\underset{\uparrow\ \text{const}}{E}\,t + S(x)$$

Then $\frac{\partial S}{\partial t} = -E$

$$\Rightarrow \quad -E + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x = 0$$

or $$\left(\frac{\partial S}{\partial x}\right)^2 = 2mE - m^2\omega^2 x^2$$

$$\Rightarrow S(x) = \int \sqrt{2mE - m^2\omega^2 x^2}\,dx + C$$

or $$S(x,t) = -Et + \int \sqrt{2mE - m^2\omega^2 x^2}\,dx + C$$

Jacobi Thm: $$B = \text{const} = \frac{\partial S}{\partial E} = -t + \int \frac{m}{\sqrt{2mE - m^2\omega^2 x^2}}\,dx$$

$$= -t + \frac{1}{\omega}\sin^{-1}\left(\frac{m\omega x}{\sqrt{2mE}}\right)$$

$$\Rightarrow x = \frac{\sqrt{2mE}}{m\omega}\sin(\omega t + B))$$

Now do Kerr geodesics w/ HJ

Inverse metric $\tilde{g} = g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$

$$= \frac{-1}{\Delta\Sigma} \left[(r^2+a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi} \right]^2$$

$$+ \frac{1}{\Sigma \sin^2\theta} \left[\frac{\partial}{\partial \phi} + a \sin^2\theta \frac{\partial}{\partial t} \right]^2$$

$$+ \frac{\Delta}{\Sigma} \left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{\Sigma} \left(\frac{\partial}{\partial \theta} \right)^2$$

HJ: $$\frac{-\partial S}{\partial \lambda} = H\left(x^\alpha, \frac{\partial S}{\partial x^\beta}\right) = \frac{1}{2} g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta}$$

$$\Rightarrow -\frac{dS}{\partial \lambda} = \frac{-1}{2\Delta\Sigma} \left[(r^2+a^2) \frac{\partial S}{\partial t} + a \frac{\partial S}{\partial \phi} \right]^2 + \frac{1}{2\Sigma \sin^2\theta} \left[\frac{\partial S}{\partial \phi} + a \sin^2\theta \frac{\partial S}{\partial t} \right]^2$$

$$+ \frac{\Delta}{2\Sigma} \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{2\Sigma} \left(\frac{\partial S}{\partial \theta} \right)^2$$

Separation of variables:

Assume $$S = \underbrace{+\frac{1}{2}\mu^2\lambda}_{\frac{\partial S}{\partial \lambda} = -H} \underbrace{-Et}_{\frac{\partial S}{\partial t} = P_t = E} \underbrace{+L\phi}_{\frac{\partial S}{\partial \phi} = P_\phi = L} + S_r(r) + S_\theta(\theta)$$

First 3 terms possible to separate trivially

b.c. λ, t, ϕ do not appear in H

So $-\dfrac{\partial S}{\partial \lambda} = H \Rightarrow$

$$-\frac{1}{2}\mu^2 \Sigma = \overbrace{-\frac{1}{2\Delta}\left[(r^2+a^2)E - aL\right]^2}^{①} + \overbrace{\frac{1}{2\sin^2\theta}\left[L - aE\sin^2\theta\right]^2}^{②}$$

$$+ \frac{\Delta}{2}\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2$$

① depends only on r

② depends only on θ use $\Sigma = r^2 + a^2\cos^2\theta$ to separate LHS

$$\Rightarrow \underbrace{-\frac{1}{2}\mu^2 r^2 + \frac{1}{2\Delta}\left[(r^2+a^2)E - aL\right]^2 - \frac{\Delta}{2}\left(\frac{\partial S_r}{\partial r}\right)^2}_{\text{depends only on } r \;=\; \frac{1}{2}K} = \underbrace{\frac{1}{2}\mu^2 a^2\cos^2\theta + \frac{1}{2\sin^2\theta}\left[L - aE\sin^2\theta\right]^2 + \frac{1}{2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2}_{\text{depends only on } \theta \;=\; \frac{1}{2}K}$$

note $K \geq 0$

$K =$ constant.   Carter constant

$$\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 = \frac{1}{\Delta}\left[(r^2+a^2)E - aL\right]^2 - \mu^2 r^2 - K \equiv \frac{R(r)}{\Delta(r)}$$

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = \frac{-1}{\sin^2\theta}\left[L - aE\sin^2\theta\right]^2 - \mu^2 a^2\cos^2\theta + K \equiv \Theta(\theta)$$

(Sometimes $Q \equiv K - (L - aE)^2$ is called Carter const.)

So $S_r = \int \frac{1}{\Delta}\sqrt{R}\,dr$ $\qquad S_\theta = \int \sqrt{\Theta}\,d\theta$

$$\Rightarrow \boxed{S = \frac{1}{2}\mu^2\lambda - Et + L\phi + \int \frac{\sqrt{R}}{\Delta}\,dr + \int \sqrt{\Theta}\,d\theta}$$

<u>Jacobi Thm</u>

$(R = (\,) - \Delta K)$ $\qquad (\Theta = (\,) + K)$

$$\text{const} = \frac{\partial S}{\partial K} = \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}(-\Delta)\,dr + \frac{1}{2}\int \frac{d\theta}{\sqrt{\Theta}}$$

$$\Rightarrow \boxed{\int^r \frac{1}{\sqrt{R}}\,dr = \int^\theta \frac{1}{\sqrt{\Theta}}\,d\theta + C} \quad (A)$$

$(R = (\,) - \Delta\mu^2 r^2)$ $\qquad (\Theta = (\,) - \mu^2 a^2 \cos^2\theta)$

$$\text{const} = \frac{\partial S}{\partial \mu^2} = \frac{1}{2}\lambda + \frac{1}{2}\int \frac{(-\Delta)\,r^2}{\Delta\sqrt{R}}\,dr + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}(-a^2\cos^2\theta)\,d\theta$$

$$\Rightarrow \boxed{\lambda + \text{const} = \int^r \frac{r^2\,dr}{\sqrt{R}} + \int^\theta \frac{a^2\cos^2\theta\,d\theta}{\sqrt{\Theta}}} \quad (B)$$

Take $\frac{d}{d\lambda}((A)) \Rightarrow \frac{1}{\sqrt{R}}\frac{dr}{d\lambda} = \frac{1}{\sqrt{\Theta}}\frac{d\theta}{d\lambda}$

$\frac{d}{d\lambda}((B)) \Rightarrow 1 = \frac{r^2}{\sqrt{R}}\frac{dr}{d\lambda} + \frac{a^2\cos^2\theta}{\sqrt{\Theta}}\frac{\partial\theta}{\partial\lambda}$

$$\Rightarrow \boxed{\begin{aligned}\Sigma\frac{dr}{d\lambda} &= \sqrt{R}\\ \Sigma\frac{d\theta}{d\lambda} &= \sqrt{\Theta}\end{aligned}}$$

($\frac{\partial S}{\partial L} = \text{const}$ + $\frac{\partial S}{\partial E} = \text{const}$ give $\frac{\partial\phi}{\partial\lambda}$ + $\frac{\partial t}{\partial\lambda}$ Eqs from before)

More on Carter const

$$Q = K - (L + aE)^2$$ is generated by Killing Tensor

let $K^{\alpha\beta} = 2\Sigma \, l^{\alpha} n^{\beta} + r^2 g^{\alpha\beta}$

where $l^{\alpha} = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right)$

$$n^{\alpha} = \left(\frac{r^2+a^2}{2\Sigma}, \frac{-\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma} \right)$$

Then $\nabla_{(\alpha} K_{\beta\gamma)} = 0$ Killing Tensor $\left(\begin{array}{c} \text{Like } \nabla_{(\beta} \xi_{\beta)} = 0 \\ \text{For } \xi_{\beta} = \text{KV} \end{array} \right)$

Turns out $Q = K^{\alpha\beta} u_{\alpha} u_{\beta}$ $\Rightarrow$ conserved along geodesic

---

For $a \to 0$ $\quad Q = P_{\theta}^2 + \frac{L^2}{\sin^2\theta}$ "total ang momentum"