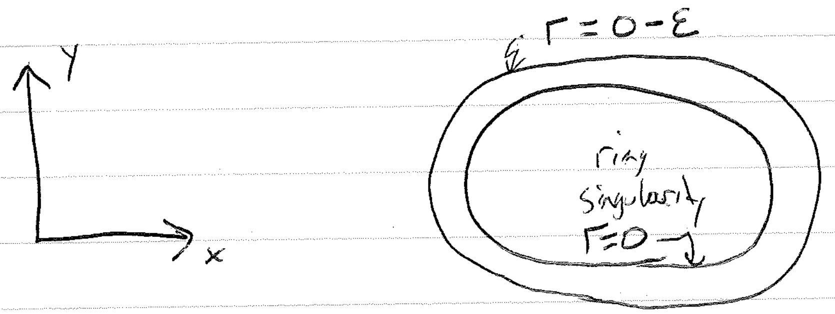
More wierdness

Consider closed curve just outside ring, $r<0$

$\theta = \pi/2$

$0 \leq \tilde{\phi} \leq 2\pi$

$t = \text{const}$



This curve can be reached by observer who goes thru the ring to $r<0$ region

$$ds^2 = g_{\tilde{\phi}\tilde{\phi}} U^{\tilde{\phi}} U^{\tilde{\phi}} \qquad \text{bc } r = -\varepsilon = \text{const}$$

$$t, \theta = \text{const}$$

but

$$g_{\tilde{\phi}\tilde{\phi}} = \frac{1}{\Sigma}\left((r^2+a^2)^2 - \Delta a^2\right)$$

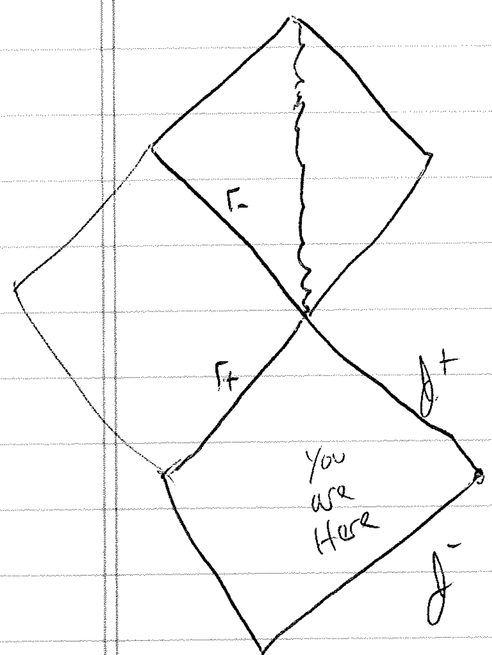
$$= \frac{1}{\Sigma}\left(r^4 + r^2 a^2 + 2Ma r\right)$$

This is <u>negative</u> for $r = -\varepsilon$

so $ds^2 < 0$

Closed curve is <u>timelike</u>

All CTCs + singularities inside Cauchy horizon



- Assumes pure vacuum, eternal BH
- Unstable to perturbations (still under investigation)
- Cauchy Horizon: singularity affects solution

So quantum gravity effects near singularity may change interior

Orbits in Kerr spacetime

$\frac{\partial}{\partial t}$ + $\frac{\partial}{\partial \phi}$ are KVs $\Rightarrow$ P_t and P_ϕ const along geodesics.

$$\text{Const} = P_t \equiv -E = g_{tt} P^t + g_{t\phi} P^\phi$$

$$= -\left(1 - \frac{2Mr}{\Sigma}\right) \frac{dt}{d\lambda} - \frac{2Mar}{\Sigma} \sin^2\theta \frac{d\phi}{d\lambda}$$

$$\text{Const} = P_\phi \equiv L = g_{\phi\phi} P^\phi + g_{\phi t} P^t$$

$$= \frac{-2Mar}{\Sigma} \sin^2\theta \frac{dt}{d\lambda} + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{d\lambda}$$

Can solve for $\frac{d\phi}{d\lambda}$ + $\frac{dt}{d\lambda}$

$$\Rightarrow \boxed{\begin{aligned} \Sigma \frac{d\phi}{d\lambda} &= \frac{2Mr}{\Delta} aE + \frac{L(\Sigma - 2Mr)}{\Delta \sin^2\theta} \\ \Sigma \frac{dt}{d\lambda} &= \frac{-2Mr}{\Delta} aL + \frac{E\left[(r^2+a^2)\Sigma + 2Mra^2\sin^2\theta\right]}{\Delta} \end{aligned}}$$

rest mass

We also know $-\mu^2 = p_\mu p^\mu = \text{const}$

$$\Rightarrow -\mu^2 = -\left(1-\frac{2Mr}{\Sigma}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{4Mar\sin^2\theta}{\Sigma}\frac{dt}{d\lambda}\frac{d\phi}{d\lambda}$$

$$+ \frac{\Sigma}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 + \Sigma\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2$$

If we knew $\frac{d\theta}{d\lambda}$, could substitute for $\frac{dt}{d\lambda}$, $\frac{d\phi}{d\lambda}$

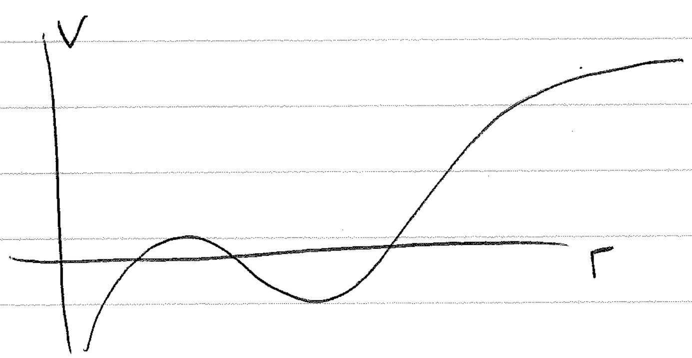
$\Rightarrow$ get eqn for $\frac{dr}{d\lambda}$

Let's do simpler case first: Equatorial orbits: $\theta = \pi/2$, $\frac{d\theta}{d\lambda} = 0$

$$\Rightarrow \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = 0$$

Effective potential

$$\boxed{V(r) = -\frac{\mu^2 M}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\mu^2 - E^2)\left(1+\frac{a^2}{r^2}\right) - \frac{M}{r^3}(L-aE)^2}$$



depends on L, E, μ, a, M

$V=0$ means $\frac{dr}{d\lambda}=0$ turning pt.

$V=0$ and $V'=0$ means circular orbit

Can solve $V=V'=0$ for circular orbit for $L+E$

You get:

$$\boxed{\begin{aligned}\frac{E}{\mu} &= \frac{r^{3/2}-2Mr^{1/2}\pm aM^{1/2}}{r^{3/4}\left(r^{3/2}-3Mr^{1/2}\pm 2aM^{1/2}\right)^{1/2}}\\ \frac{L}{\mu} &= \frac{\pm M^{1/2}\left(r^2\mp 2aM^{1/2}r^{1/2}+a^2\right)}{r^{3/4}\left(r^{3/2}-3Mr^{1/2}\pm 2aM^{1/2}\right)^{1/2}}\end{aligned}}$$

Circ. orbit. $\theta=\pi/2$

$+$ = corotating

$-$ = counter rotating

You also get $\boxed{\Omega=\frac{d\phi}{dt}=\frac{\pm M^{1/2}}{r^{3/2}\pm aM^{1/2}}}$

For circular orbit, need $r^{3/2}-3Mr^{1/2}\pm 2aM^{1/2}\geq 0$ denom of E/μ

equality when $r=r_{ph}=2M\left(1+\cos\left(\frac{2}{3}\cos^{-1}(\mp a/M)\right)\right)$

photon orbit $(E/\mu\to\infty)$

$$r_{ph}=\begin{cases}3M & a=0\\ M & a=M,\ +\\ 4M & a=M,\ -\end{cases}$$

All circ orbits have $r>r_{ph}$

Orbit is bound when $E/\mu < 1$ perturb. causes particle to go to ∞.

Not all $r > r_{ph}$ circ orbits are bound.

bound if $E/\mu \leq 1 \Rightarrow \boxed{r > r_{mb} = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2}}$

$$r_{mb} = \begin{cases} 4M & a = 0 \\ M & a = M, + \\ 5.8M & a = M, - \end{cases}$$

Orbit is stable if $V''(r) > 0$

Can show $V''(r) > 0 \Rightarrow 1 - (E/\mu)^2 > \frac{2}{3} M/r$ or $r > r_{isco}$

where $r_{isco} = M\left(3 + Z_2 \mp \left((3 - Z_1)(3 + Z_1 + 2Z_2)\right)^{1/2}\right)$

where $Z_1 \equiv 1 + (1 - a^2/M^2)^{1/3}\left[(1 + a/M)^{1/3} + (1 - a/M)^{1/3}\right]$

$Z_2 \equiv \left(3a^2/M^2 + Z_1^2\right)^{1/2}$

$$r_{isco} = \begin{cases} 6M & a = 0 \\ M & a = M, + \\ 9M & a = M, - \end{cases}$$

as $a \to M$ $r_{ph} < r_{mb} < r_{isco}$

all 3 are distinct

(non zero proper distance)

Note: For $a = M(1-\delta)$

$$r_{ISCO} \approx M\left(1+(4\delta)^{1/3}\right)$$

What about non-equatorial orbits?

Use Hamilton-Jacobi.

Recall $L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$ is a valid Lagrangian for geodesic eqn.

$L = L(\dot{x}^\alpha, x^\alpha)$

i.e. $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}$ gives geodesics

Euler-Lagrange

Lagrangian $\rightarrow$ Hamiltonian let momentum $P_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = g_{\alpha\beta}\dot{x}^\beta$

Then $H = P_\alpha \dot{x}^\alpha - L$

$= P_\alpha g^{\alpha\beta} P_\beta - \frac{1}{2} g^{\alpha\beta} P_\alpha P_\beta$

$= \frac{1}{2} g^{\alpha\beta} P_\alpha P_\beta$ note $H = -\frac{1}{2}\mu^2$. $\mu =$ rest MASS

So $H = H(x^\alpha, P_\alpha)$

Hamilton's eqs $\Rightarrow$

$$\boxed{\begin{aligned} \frac{dx^\alpha}{d\lambda} &= \frac{\partial H}{\partial P_\alpha} \\ \frac{dP_\alpha}{d\lambda} &= -\frac{\partial H}{\partial x^\alpha} \end{aligned}}$$

Hamilton-Jacobi method

Consider action $S = \int L \, d\lambda$ $\qquad S = S(\lambda, x^\alpha)$

Then $P_\alpha = \frac{\partial S}{\partial x^\alpha}$

So consider H a function of x^α + $\frac{\partial S}{\partial x^\beta}$

$$H = H\left(x^\alpha, \frac{\partial S}{\partial x^\beta}\right)$$

Note $L = \frac{dS}{d\lambda} = \frac{\partial S}{\partial \lambda} + \frac{\partial S}{\partial x^\alpha}\dot{x}^\alpha$

$= \frac{\partial S}{\partial \lambda} + P_\alpha \dot{x}^\alpha$ $\qquad$ So $\frac{\partial S}{\partial \lambda} = L - P_\alpha \dot{x}^\alpha$

$= -H$

$$\Rightarrow \boxed{\frac{\partial S}{\partial \lambda} + H\left(x^\alpha, \frac{\partial S}{\partial x^\beta}\right) = 0}$$

Hamilton-Jacobi Equation

PDE for $S(\lambda, x^\alpha)$

Solution S will depend on non additive N integration constants α_n

Jacobi Theorem $\qquad \frac{\partial S}{\partial \alpha_n} \equiv \beta^n$ is also a constant.

Also, if H indep of some x^1 then $S = \alpha_1 x^1 + S(x^2, x^3, \ldots, \lambda)$

+ $\quad P_1 = \frac{\partial S}{\partial x_1} = \text{const}$