Let's remove coord singularity at horizon.

Kerr coordinates: analogue of Eddington - Finkelstein

Recall: EF coords chosen so \( \tilde{\nu} \) const along ingoing null rays,

Kerr coords: choose new coords \( \tilde{\nu}, \tilde{\Phi} \) const along ingoing null rays

In Kerr, (Kerr coords), ingoing null rays have tangent

\[ l^\mu = \left( \frac{r^2 + a^2}{\Delta}, -1, 0, \frac{a}{\Delta} \right) \]

From dragging of photons

radial at \( \infty \)

\[ \frac{\Delta}{\Delta} \]

So

\[ \frac{dt}{dr} = -\frac{r^2 + a^2}{\Delta}, \quad \frac{d\Phi}{dr} = \frac{-a}{\Delta} \]

Choose

\[ d\tilde{\nu} = dt + \frac{r^2 + a^2}{\Delta} \, dr \]

like EF

\[ d\tilde{\Phi} = d\Phi + \frac{a}{\Delta} \, dr \]

"Unwinding"

Then \( \frac{d\tilde{\nu}}{dr} = \frac{d\tilde{\Phi}}{dr} = 0 \) for these geodesics
\[ ds^2 = -(1-\frac{2Mr}{\varepsilon})d\tilde{\omega}^2 + 2d\tilde{\omega}d\tilde{r} + \Sigma d\theta^2 \]

\[ + \frac{(r^2+a^2)^2-\Delta a^2 \sin^2 \phi}{\Sigma} \sin^2 \theta d\phi^2 \]

\[-2a \sin \phi d\tilde{\omega} \cdot \frac{4\pi r}{\Sigma} \sin \theta d\nu d\phi \]

Kerr coords

No coord. singularity at \( \Delta = 0 \)

\[ \uparrow \tilde{\omega} \]

\[ \uparrow \phi \]

Diagram showing outgoing and ingoing photons, indicating a possible event horizon or singularity.
Curvature singularity

Ra85 Ra85 ∼ \( M^2 \beta^3 \)

blows up when \( \Sigma \rightarrow 0 \)

\( \Sigma = r^2 + a^2 \cos^2 \theta \)

So \( \Sigma = 0 \) means \( r = 0 \) and \( \theta = \pm \frac{\pi}{2} \)

But in spherical coords \( r = 0 \) is a coord singularity, what is \( r = 0 \) and \( \theta \neq \frac{\pi}{2} \)?

Remove coord singularity via \( \text{Kerr-Schild coords} \)

Let \( x + iy = (r + ia)e^{i \phi} \sin \theta \)

\( z = r \cos \theta \)

\( \gamma = \sqrt{r^2 - \Gamma} \)

Equivalently

\( x = r \Gamma \sin \theta \cos (\phi + \tan^{-1} \frac{\alpha}{r}) \)

\( y = r \Gamma \sin \theta \sin (\phi + \tan^{-1} \frac{\alpha}{r}) \)

\( z = r \cos \theta \)

\( \gamma = \sqrt{r^2 - \Gamma} \)

looks like spheroidal coords.

\( \Gamma, \Theta \) considered functions of \( x, y, \gamma \)

\[ \frac{x^2 + y^2}{a^2 \sin^2 \theta} - \frac{z^2}{a^2 \cos^2 \theta} = 1 \]

\[ 2\Gamma^2 = \frac{x^2 + y^2 \gamma^2 - a^2}{\Gamma^2 + a^2} + N(x^2 + y^2 \gamma^2 - a^2) + \gamma^2 a^2 \]

\[ \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \]
Kerr-Schild metric:

\[ ds^2 = -\tilde{d}T^2 + dx^2 + dy^2 + dz^2 \]

\[ + \frac{2M r^3}{r^4 + a^2 z^2} \left[ d\tilde{E} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} \right]^2 \]

\text{where} \ a \neq \text{cong}\]

Note that \( g_{\mu\nu} = \bar{g}_{\mu\nu} + H \delta_{\mu\nu} \)

\text{Flat}

\[ H = \frac{2Mr^3}{r^4 + a^2 z^2} \]

\[ \tilde{x} \times \tilde{y} \times \tilde{z} \]

\[ \bar{\mu} = \left( -1, \frac{r x + a y}{r^2 + a^2}, \frac{r y - a x}{r^2 + a^2}, \frac{z}{r^2 + a^2} \right) \]

Can show \( \bar{\mu} \) is the same null vector (different coords) used to define

Kerr coords previously: incoming null geodesic, radial at \( \infty \).

(Important for numerical relativity)
\[ \Gamma = \text{const} \text{ surfaces are ellipsoids} \]

\[ \Gamma = 0 \text{ is a disk: } x^2 + y^2 \leq a^2, \quad z = 0 \]

\[ x^2 + y^2 = a^2 \sin^2 \theta \]

\[ \theta = \frac{\pi}{3} \text{ is } x^2 + y^2 = r^2 + a^2, \quad z = 0 \]

an annulus.

\[ \Gamma = 0 \text{ and } \theta = \frac{\pi}{2} \text{ is a ring } x^2 + y^2 = a^2, \quad z = 0 \]

ring singularity

metric nonsingular off of ring
\[ \Gamma = 0 \text{ is a disk } \quad x^2 + y^2 \leq a^2, \quad z = 0 \]

\[ (x^2 + y^2 = a^2 \cos^2 \theta) \]

\[ \theta = \pi/2 \text{ is } x^2 + y^2 = r^2 + a^2, \quad z = 0 \quad \text{annulus} \]

Singularity is ring \[ x^2 + y^2 = a^2, \quad z = 0 \]
Notice - Metric nonsingular at $\Gamma = 0$, $\theta \neq \frac{\pi}{2}$

- Two solutions to $\Gamma^2 = x^2 + y^2 + z^2 - a^2 + \sqrt{(x^2 + y^2 + a)^2 + y^2 z^2}$

$\Gamma > 0$ and $\Gamma < 0$

Can continue metric thru $\Gamma = 0$

Consider observer at $\theta = 0$, $x = y = 0$, $z = \Gamma$, falling thru origin

$\Gamma > 0$

Observer emerges w/ $\Gamma < 0$, $\theta = 0$ can escape to $\Gamma \rightarrow -\infty$

(no horizon for $\Gamma < 0$)

Similarly, $\frac{d\Gamma}{d\lambda}$ and $\theta$ are continuous

Curvature scalars are continuous
Penrose Diagram for Kerr

- Multiple regions
  - $\Gamma < 0$ regions have no horizon but are subject to naked Cauchy horizons
- $\Gamma > 0$ regions have Cauchy horizons

$\text{etc.}$