Let's remove coord singularity at horizon.

Kerr coordinates: analogue of Eddington-Finkelstein

Recall: EF coords chosen so $\tilde{v}$=const along ingoing radial null rays.

Kerr coords: choose new coords $\tilde{v}, \tilde{\phi}$ const along ingoing null rays

In Kerr, (BL coords), choose ingoing null rays that have tangent

$$\ell^\mu = \left(\frac{r^2+a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right)$$

Frame dragging of photons

radial at ∞ const θ

$$\text{so } \frac{dt}{dr} = \frac{-r^2+a^2}{\Delta}, \quad \frac{d\phi}{dr} = \frac{-a}{\Delta}$$

Choose $d\tilde{v} = dt + \frac{r^2+a^2}{\Delta} dr$ like EF

$d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr$ "Unwinding"

Then $\frac{d\tilde{v}}{dr} = \frac{d\tilde{\phi}}{dr} = 0$ for these geodesics

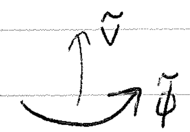
$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)d\tilde{v}^2 + 2\,d\tilde{v}\,dr + \Sigma\, d\theta^2$$

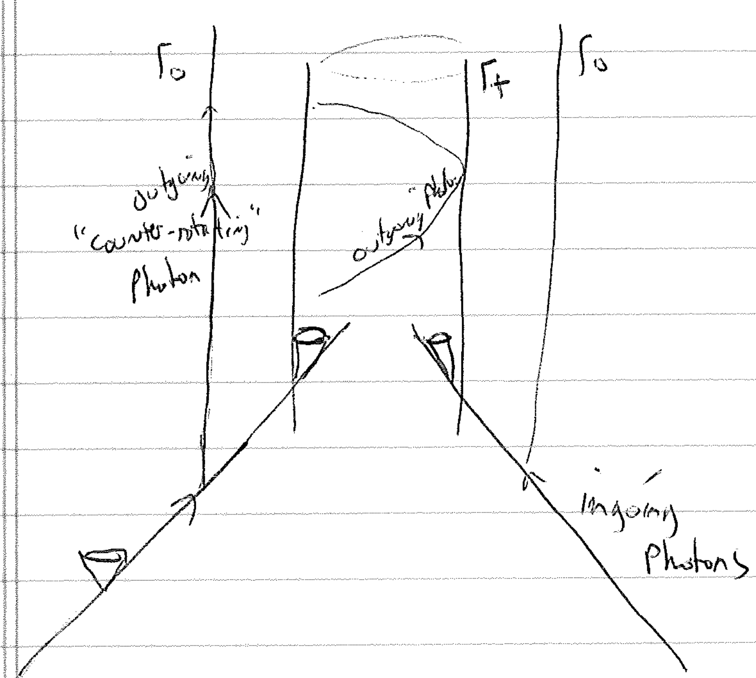
$$+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta\, d\tilde{\phi}^2$$

$$-2a\sin^2\theta\, dr\, d\tilde{\phi} - \frac{4Mar}{\Sigma}\sin^2\theta\, dv\, d\tilde{\phi}$$

Kerr coords

No coord singularity at $\Delta = 0$





Curvature singularity

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim M^2/\Sigma^3$$

blows up when $\Sigma \to 0$ $\quad \Sigma = r^2 + a^2\cos^2\theta$

so $\Sigma = 0$ means $r=0$ <u>and</u> $\theta = \pm\pi/2$

But in spherical coords $r=0$ is a coord singularity. What is $r=0$ and $\theta \neq \pm\pi/2$?

Remove coord singularity via <u>Kerr-Schild coords</u>

Let
$$x + iy = (r+ia)e^{i\tilde{\phi}}\sin\theta$$
$$z = r\cos\theta$$
$$\bar{t} = \tilde{V} - r$$

$\tilde{\phi}, \tilde{V}$ = Kerr coords

Equivalently
$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\left(\tilde{\phi} + \tan^{-1} a/r\right)$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\left(\tilde{\phi} + \tan^{-1} a/r\right)$$
$$z = r\cos\theta$$
$$\bar{t} = \tilde{V} - r$$

looks like spheroidal coords.

<u>r, θ considered functions of x, y, z</u>

$$2r^2 = x^2+y^2+z^2-a^2 + \sqrt{(x^2+y^2+z^2-a^2)^2 + 4a^2z^2}$$

Note
$$\frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1$$

&
$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$$

Kerr – Schild metric:

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2$$

$$+ \underset{\text{bad at ring}}{\frac{2Mr^3}{r^4 + a^2 z^2}} \Bigg[\underbrace{d\tilde{t} + \frac{r(x\,dx + y\,dy) - a(x\,dy - y\,dx)}{r^2 + a^2} + \frac{z\,dz}{r}}_{\text{OK}} \Bigg]^2$$

Note that $\quad g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Flat}} + H l_\mu l_\nu$

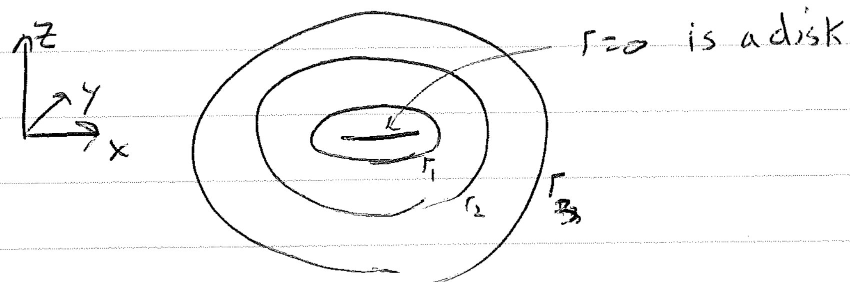
where $\quad H \equiv \dfrac{2Mr^3}{r^4 + a^2 z^2}$

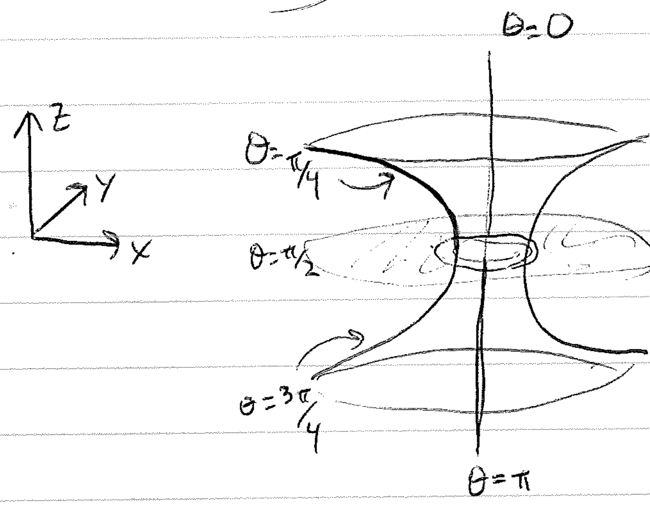
and
$$l_\mu = \begin{array}{c} \begin{array}{cccc} \tilde{t} & x & y & z \end{array} \\ \left(-1, \dfrac{rx + ay}{r^2 + a^2}, \dfrac{ry - ax}{r^2 + a^2}, \dfrac{z}{r} \right) \end{array}$$

Can show $\vec{l}$ is the same null vector (different coords) used to define Kerr coords previously: ingoing null geodesic, radial at ∞.

(Important for numerical relativity)

$r = \text{const}$ surfaces are ellipsoids





$r=0$ is a disk : $x^2+y^2 \le a^2$, $z=0$ $\qquad x^2+y^2 = a^2\sin^2\theta$

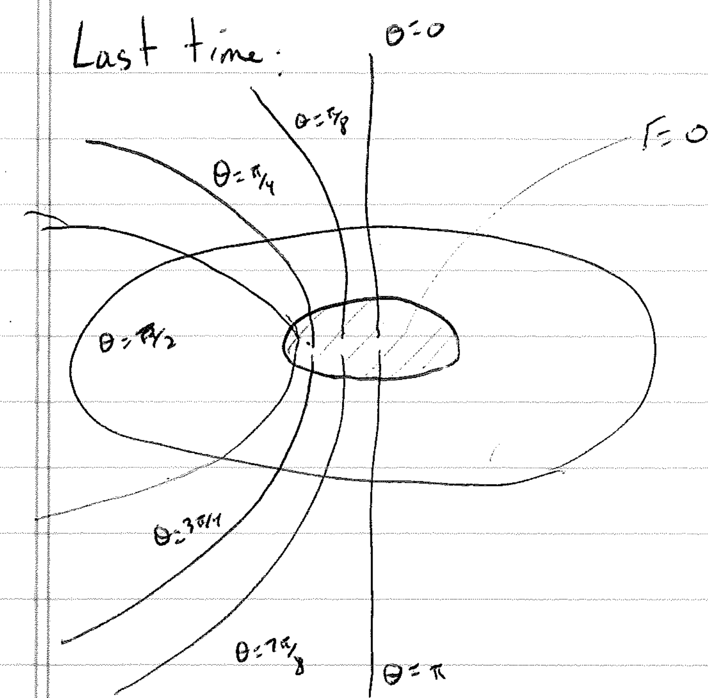
$\theta = \pi/2$ is $x^2+y^2 = r^2+a^2$, $z=0$

an annulus.

$r=0$ and $\theta = \pi/2$ is a <u>ring</u> $x^2+y^2=a^2$, $z=0$

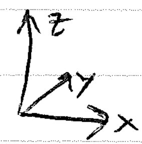
<u>ring singularity</u> metric nonsingular off of ring

Last time:



Kerr-Schild coords

x, y, z



$r = 0$ is a disk $\quad x^2 + y^2 \le a^2, \; z = 0$

$$\downarrow$$

$$\left(x^2 + y^2 = a^2 \sin^2\theta \right)$$

$\theta = \pi/2$ is $\quad x^2 + y^2 = r^2 + a^2, \; z = 0$

annulus

Singularity is ring $\quad x^2 + y^2 = a^2, \; z \equiv 0$

Notice - Metric nonsingular at $r=0$, $\theta \neq \pi/2$

- two solutions to $r^2 = x^2+y^2+z^2-a^2+\sqrt{(x^2+y^2+z^2-a^2)^2+4a^2z^2}$

$r>0$ and $r<0$

Can continue metric thru $r=0$

Consider observer at $\theta=0$, $x=y=0$, $z=r$, falling thru origin

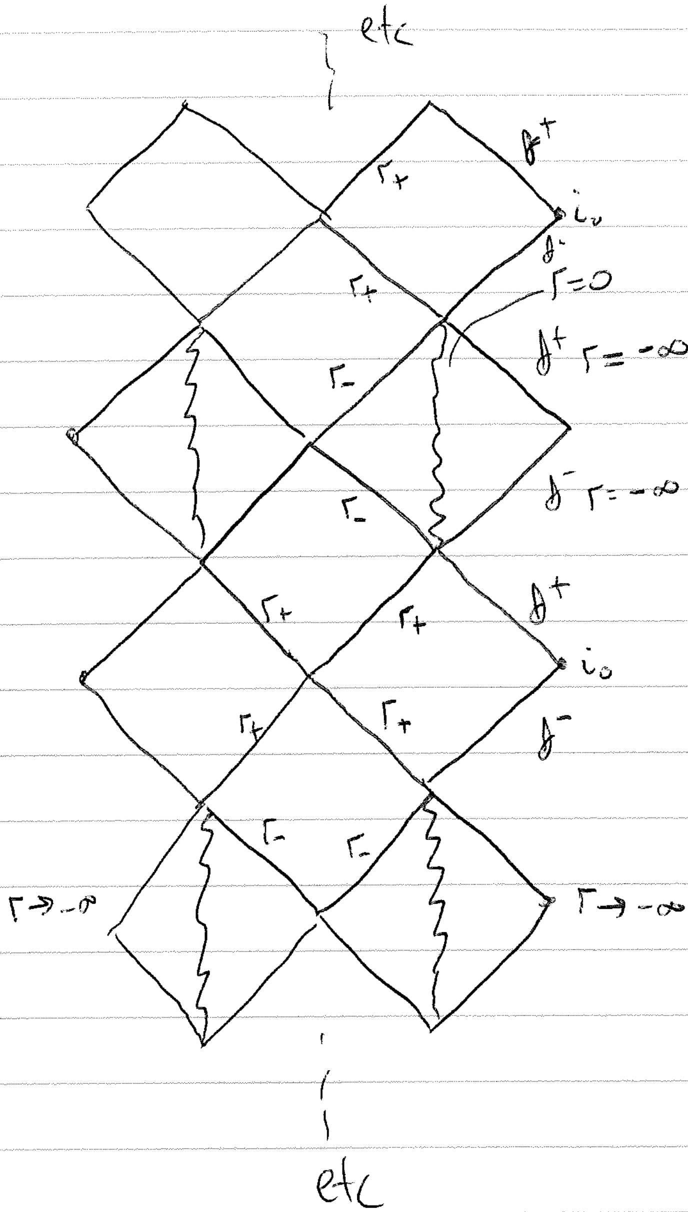
observer emerges w/ $r<0$, $\theta=0$ can escape to $r\to-\infty$

(no horizon for $r<0$)

~~Similarly~~, $\frac{dr}{d\lambda}$ and θ are continuous

Curvature scalars are continuous

Penrose Diagram for Kerr



- Multiple regions.
- $r<0$ regions have no horizon but naked singularity
- Cauchy horizons