Rotating black holes

- Generically, astrophysical BHs will have angular momentum.

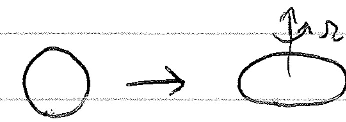
- Described in GR by Kerr solution

- Kerr much more difficult + complicated than Schwarzschild

  GR (1915) Schwarzschild (1915) Kerr (1963)!

Derive Kerr solution (From Teukolsky, arXiv:1410.2130)

- In Newtonian gravity, spin "flattens" stars to oblate spheroids



(= ellipsoid w/ two equal axes > third axis)

An oblate spheroid has Cartesian coords

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$
$$z = r\cos\theta$$

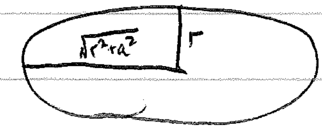
for $r = \text{const}$, $a = \text{const}$

(note $r^2 \neq x^2+y^2+z^2$)

minor axis has length $2r$

major axes have length $2\sqrt{r^2+a^2}$

as $a \to 0$, spheroid $\to$ sphere



For const "a", each point in 3D space is on a spheroid with some value of r.

So use r, θ, ϕ as coordinates "oblate spheroidal coords"

Write Flat space metric in terms of these coords

$$ds^2_{Flat} = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= -dt^2 + \frac{\Sigma}{r^2+a^2}\, dr^2 + \Sigma\, d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2$$

where $\Sigma \equiv r^2 + a^2\cos^2\theta$

Can express this as

$$ds^2_{Flat} = -\frac{(r^2+a^2)}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\Sigma}{r^2+a^2}\, dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\,d\phi - a\,dt\right)^2$$

Note; $d\phi\, dt$ terms cancel

but they won't cancel if you have rotation

So make ansatz to add rotation:

- In 1st term, let $r^2+a^2 \rightarrow r^2+a^2 - Z(r)$ ← arbitrary function so $d\phi\, dt$ terms don't cancel

also, in 2nd term, g_{rr} should depend on M when $a \to 0$ (Schwarzschild)

So replace $r^2+a^2 \rightarrow F(r) + r^2 + a^2$ ← arb. function that depends on M

Made unjustified assumption that a = oblate distortion param is same as spin, will see it is true later.

Then $$ds^2 = -\frac{(r^2+a^2-Z(r))}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\Sigma}{r^2+a^2+F(r)}\, dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\,d\phi - a\,dt\right)^2$$

or $$ds^2 = -dt^2 + \frac{\Sigma}{r^2+a^2+F(r)}\, dr^2 + \Sigma\, d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 + \frac{Z(r)}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2$$

Now substitute into $G_{\mu\nu} = 0$ and try to solve for $F(r)$ + $Z(r)$.

$$G_{rr} = 0 \quad \Rightarrow \quad \underbrace{(\quad\quad)}_{\text{depends on } r \text{ but not } \theta} + \underbrace{(\quad\quad)}\cos^2\theta = 0$$

$\Rightarrow$ get 2 eqns

$$\left[Z + r(r - Z')\right]F - r(r^2+a^2)Z' + (2r^2+a^2)Z = 0$$

$$r(Z'-r)F + Z^2 + r(r^2+a^2)Z' - (2r^2+a^2)Z = 0$$

Add eqs $\Rightarrow$ $Z(F+Z) = 0$

either $Z = 0$ (leads to $F = 0$) (recover flat metric)

or $\boxed{F = -Z}$

Subst. $F = -Z$: $(r^2+a^2-Z)(rZ'-Z) = 0$

$r^2+a^2 = Z$ means $F = -(r^2+a^2)$ and $g_{rr} \to \infty$ singular.

So choose $rZ' - Z = 0 \Rightarrow \boxed{Z = Cr}$ ← constant

So $\boxed{F = -Cr}$

Find constant $\Rightarrow$ but for $a = 0$ (no spin), should have Schwarzschild,

so $g_{rr} = \dfrac{r^2}{r^2 - Cr}$ should be $\left(1 - \dfrac{2M}{r}\right)^{-1}$

so $C = 2M$

Spinning BH

Kerr (1963) solution:

$$\boxed{ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \left(r^2+a^2+\frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2}$$

or (same thing)

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)d\phi - a\,dt\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

Where $\Delta \equiv r^2 - 2Mr + a^2$

$\Sigma \equiv r^2 + a^2\cos^2\theta$ (Σ is sometimes called ρ^2)

$a \equiv S/M$ ang momentum / unit mass

t, r, θ, ϕ = Boyer–Lindquist coords

Properties of Kerr

- Depends on only 2 params: a & M

- $a=0 \Rightarrow$ Schwarzschild so M is mass

- Metric comps. indep of t + ϕ

$$\Rightarrow \quad \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} \quad \text{are KVs}$$

only KVs

- Discrete symmetry under $t \to -t$
  and $\phi \to -\phi$

<u>stationary</u> but not <u>static</u>.

spacetime rotates about z axis

- Asymptotically Flat

- For $M \to 0$ $(a \neq 0)$

$$\Rightarrow ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

where
$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Flat space in spheroidal coords

Kerr horizons

Horizon when $r = \text{const}$ surface becomes null $\quad (g^{rr} = 0)$

$$\Rightarrow \Delta = 0 = r^2 + a^2 - 2Mr$$

Two solutions:

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

$|a| \le M$:

r_+ = outer horizon = event horizon

r_- = inner horizon = Cauchy horizon $\quad \Rightarrow$ Black hole solution

$|a| = M \quad$ "extremal" Kerr. $\quad r_+ = r_-$

$|a| > M \quad r_+, r_-$ complex

No horizon, no black hole

From now on consider only $|a| \le M$.

Can write $\Delta = (r - r_-)(r - r_+)$

Stationary, static, and ZAMO observers

Two Killing Vectors $\frac{\partial}{\partial \phi}$ + $\frac{\partial}{\partial t}$

$\Rightarrow$ Geodesics have 2 constants of motion

$$E \equiv \vec{P} \cdot \frac{\partial}{\partial t} \quad , \qquad L \equiv \vec{P} \cdot \frac{\partial}{\partial \phi}$$

Consider observer with zero angular momentum $L = 0$

ZAMO   Not necessarily on a geodesic

The angular velocity as seen from ∞ is $\Omega = \frac{d\phi}{dt} = \frac{U^\phi}{U^t}$

For ZAMO, $L = m U_\phi = 0$

$$\Rightarrow \quad U_\phi = g_{\phi\alpha} U^\alpha = g_{\phi\phi} U^\phi + g_{\phi t} U^t = 0$$

$$\Rightarrow \quad \frac{U^\phi}{U^t} = \frac{-g_{\phi t}}{g_{\phi\phi}}$$

$$\text{so } \Omega = \frac{-g_{\phi t}}{g_{\phi\phi}} = \frac{2Mar}{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}$$

- $\Omega \to 0$ as $r \to \infty$ ZAMO nonrotating at ∞

as you expect if $L = 0$

But if a ZAMO falls near the BH,

$\Omega \neq 0$ despite the particle having no ang momentum.

Also, $(r^2+a^2)^2 > a^2 \Delta \sin^2\theta$
$= a^2 \sin^2\theta (a^2+r^2-2Mr)$

So denom of $\Omega = \dfrac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}$ is always >0

$\Rightarrow$ Ω is in the same direction as a.

$\Rightarrow$ "Frame dragging". BH drags observers around the hole, giving them ang. velocity.

2 more special types of observers:

Static observer: r, θ, ϕ = const $\vec{u} \propto \frac{\partial}{\partial t}$

Stationary observer: r, θ = const $\vec{u} \propto \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$

$\Omega = \text{const} = \frac{d\phi}{dt}$

So for stationary observer $\begin{cases} u^\phi = \Omega u^t, \text{ so} \\ \vec{u} = u^t(1,0,0,\Omega) \end{cases}$

For static observer $\vec{u} = u^t(1,0,0,0)$

Static observer:

$$\vec{u}\cdot\vec{u} = -1 = (u^t)^2 g_{tt}$$

$$\Rightarrow \quad g_{tt} < 0$$

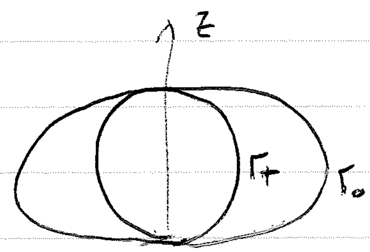
$$\Rightarrow 2Mr - \Sigma < 0$$

$$\Rightarrow r^2 - 2Mr + a^2\cos^2\theta > 0$$

$$\Rightarrow \quad r > \boxed{r_0 = M + \sqrt{M^2 - a^2\cos^2\theta}}$$

r_0 is <u>static limit</u>.

No static observers for $r < r_0$
(no matter how they accelerate!)
Frame dragging.



Note $r_0 \geq r_+$

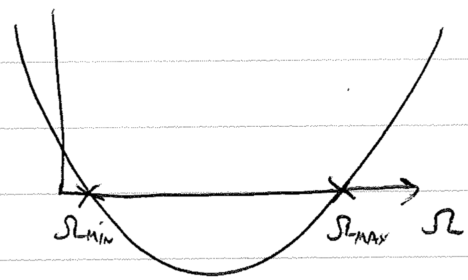
Surface $r(\theta) = r_0$
is called <u>ergosphere</u>
(see why later)
Region between r_+ + r_0 is
called <u>ergoregion</u>.

Stationary observer

$$\vec{u}\cdot\vec{u} = -1 = (u^t)^2\left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right) = -1$$

This means: $g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0$

Cannot have stationary observer for all Ω



$\Omega_{\substack{min\\max}}$ when $g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} = 0$

$$\Rightarrow \quad \Omega_{\substack{max\\min}} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}$$

No real solution unless $0 < g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta \sin^2\theta$

$\Rightarrow \quad \Delta > 0$ or $\boxed{r > r_+}$

No stationary observer inside horizon

At horizon $\Omega_{min} = \Omega_{max} = \dfrac{-g_{t\phi}}{g_{\phi\phi}}$ same as ZAMO

$= \dfrac{a}{2Mr_+}$ at $r \to r_+$

Note, particle worldline $\to$ null at $r \to r_+$

Only 1 possible ang velocity of stationary particle at r_+

Also, inside ergosphere, i.e. $r < r_0$, $g_{tt} > 0$

so $\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} < |g_{t\phi}|$

$\Rightarrow$ Ω_{max} + Ω_{min} have the same sign,

corotating w/ BH

$\Rightarrow$ Stationary observers inside $r = r_0$ must corotate w/ BH

Also, for $r \to \infty$ $\quad \Omega_{\substack{MAX \\ MIN}} \to \dfrac{\pm 1}{r \sin\theta}$