Reissner - Nordström black hole

- RN metric describes static, spherically symmetric black hole w/ mass M + electric charge Q.

- Solves Einstein + Maxwell equations (vacuum otherwise)

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega^2$$

where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

$$A_\mu = \left(\frac{Q}{r}, 0, 0, 0\right)$$

Q = charge, measured by distant particles

- For $Q = 0$, $\Rightarrow$ Schwarzschild

- Can think of effective "mass" $m_{eff}(r) = M - \frac{Q^2}{2r}$ Then RN is like Schwarzschild

- Singularities: like Schwarzschild, true singularity at $r = 0$

$$R^{\alpha\epsilon\gamma\delta} R_{\alpha\epsilon\gamma\delta} \to 0 \quad \text{as} \quad r \to 0$$

- Horizons / coordinate singularities

Like Schwarzschild, coords go bad when $f(r) = 0 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

Two roots:

$$r_+ = M + \sqrt{M^2 - Q^2}$$

$$r_- = M - \sqrt{M^2 - Q^2}$$

Three cases

① $|Q| \leq M \Rightarrow r_+$ and r_- are real, $r_+ > r_-$

Event horizon at $r = r_+$ (like Schwarzschild)

Inner horizon $r = r_-$ is inside the BH. (more later)

② $|Q| = M \Rightarrow r_+ = r_- = M$ Extremal RN black hole

③ $|Q| > M \Rightarrow$ No real solns for r_+, r_-

- $f(r)$ always $\neq 0$ for all real r.
- No horizons, no coord. singularity
- Not a black hole.

- Still have $r = 0$ physical singularity, "naked"

For all the world to see.
Not "clothed" by a horizon.

- Null radial geodesics $ds^2 = 0 = -f(r)dt^2 + \frac{1}{f(r)} dr^2$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = \left(f(r)\right)^2 \qquad \frac{dr}{dt} = \pm |f(r)|$$

Never changes sign

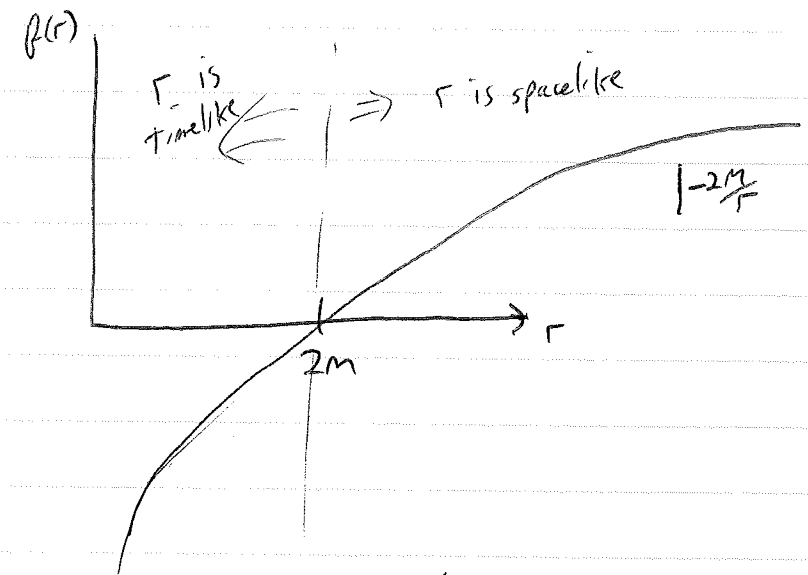
- $|Q| > M$ Thought to be unphysical

Cosmic censureship conjecture (Penrose): There are NO naked singularities

From now on, we will assume $|Q| \leq M$.

Look more at $r=0$ singularity:

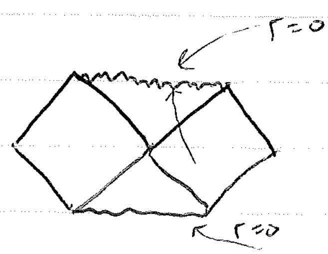
① In Schwarzschild, r is a timelike coord inside horizon.



So $r=\text{const}$ surface is spacelike inside $r=2M$

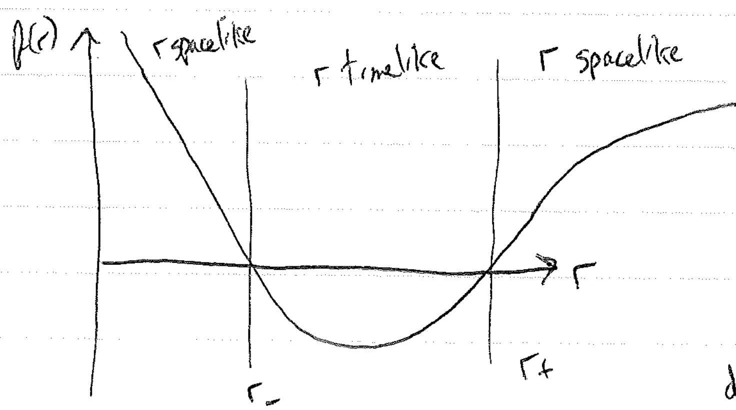
$$ds^2 = +|f(r)|\,dt^2 + r^2 d\Omega^2$$

In particular, $r=0$ is a spacelike surface in Schwarzschild



All observers inside BH must hit singularity!

② RN, $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}$



r is a spacelike coord inside $r=r_-$

So $r=\text{const}$ surface is timelike inside $r=r_-$

$$ds^2 = -|f(r)|\,dt^2 + r^2 d\Omega^2$$

In particular, as $r \to 0$, $ds^2 \to -\frac{r_+ r_-}{r^2} dt^2$

So the $r=0$ physical singularity is a timelike singularity.

Observers can avoid $r=0$!

Let's see how this works by looking at radial observers for RN

Let observer fall into BH

Use ingoing EF coords: $\tilde{v} = t + r^*$ where $dr^* \equiv \frac{dr}{f(r)}$

$$\Rightarrow ds^2 = -f(r)\, d\tilde{v}^2 + 2\, d\tilde{v}\, dr + r^2 d\Omega^2$$

metric regular at r_+, r_-, still singular at $r=0$

4-velocity $\vec{u} = \frac{d\vec{x}}{d\tau} = (\dot{\tilde{v}}, \dot{r}, 0, 0)$ where $\dot{} = \frac{d}{d\tau}$ $\tau =$ proper time

timelike Killing vector $\vec{t} = \frac{\partial}{\partial \tilde{v}} = (1, 0, 0, 0)$

Conserved energy $E = -u_\alpha t^\alpha = -u_{\tilde{v}} = -g_{\tilde{v}\tilde{v}} u^{\tilde{v}} - g_{\tilde{v}r} u^r$

$= f(r)\dot{\tilde{v}} - \dot{r}$

so $\boxed{\dot{\tilde{v}} = \frac{E + \dot{r}}{f(r)}}$ ⓐ

But $\vec{u}\cdot\vec{u} = -1 = -f(r)\dot{\tilde{v}}^2 + 2\dot{r}\dot{\tilde{v}}$ ⓑ

Solve ⓐ+ⓑ $\Rightarrow$
$$\boxed{\begin{aligned} \dot{r}_{in} &= -\sqrt{E^2 - f(r)} \\ \dot{\tilde{v}}_{in} &= \frac{E - \sqrt{E^2 - f(r)}}{f(r)} \end{aligned}}$$

assume ingoing particle

Note that all is OK when observer crosses r_+ and r_-, where $f(r)=0$

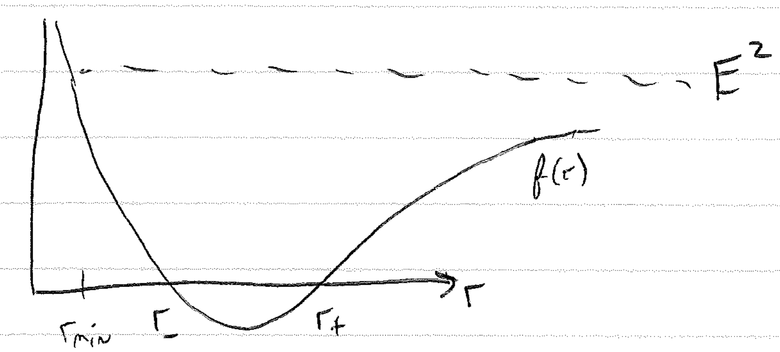
$$\Rightarrow \quad \dot{r}_{in} = -E$$

$$\dot{\tilde{V}}_{in} \approx \frac{1}{2E} \qquad \underline{\text{both Finite}}$$

Qualitatively, what happens?

$$\text{Note } \dot{r}^2 + f(r) = E^2$$

Like particle of energy E moving in potential $f(r)$



When $r = r_{min}$, particle must turn around + go <u>outward</u>. $\left(\dot{r}=0 \text{ at } r=r_{min}\right)$

turning point

So now particle moves outward, so eq. of motion is

$$\dot{r}_{out} = +\sqrt{E^2 - f(r)}$$

$$\dot{\tilde{V}}_{out} = \frac{E + \sqrt{E^2 - f(r)}}{f(r)}$$

But when outgoing observer crosses $r = r_-$

$$\dot{V}_{out} \sim \frac{2E}{f(r)} \to \infty \quad \text{as } r \to r_-$$

Ingoing EF coords break down, cannot follow observer outward.

So for outgoing observer, use outgoing EF coords

$$(\tilde{u}, r, \theta, \phi)$$

$$\tilde{u} \equiv t - r^*$$

So $$ds^2 = -f(r)\,d\tilde{u}^2 - 2\,d\tilde{u}\,dr + r^2 d\Omega^2$$

In these coords, particle with same energy E obeys

$$\dot{r}_{out} = \sqrt{E^2 - f(r)}$$

$$\dot{\tilde{u}}_{out} = \frac{E - \sqrt{E^2 - f(r)}}{f(r)}$$

radially outgoing particle

and when $f(r) = 0$, $\dot{r}_{out} = E$

$$\dot{\tilde{u}}_{out} = \frac{1}{2E}$$

ok

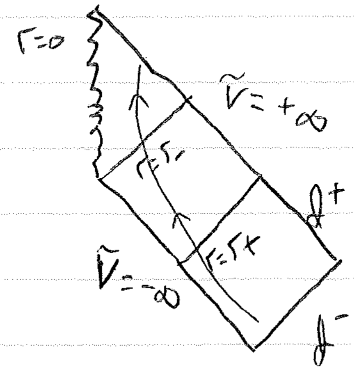
So now particle goes out, past $r = r_-$, and then out past $r = r_+$

Particle emerges from horizon! Huh?

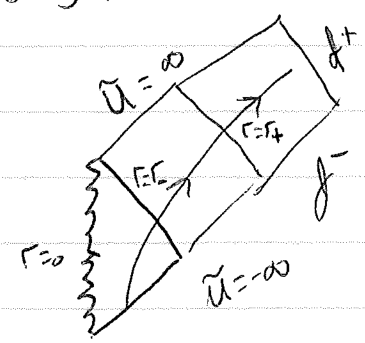
I thought nothing can get out of a BH!

Coordinate patches:

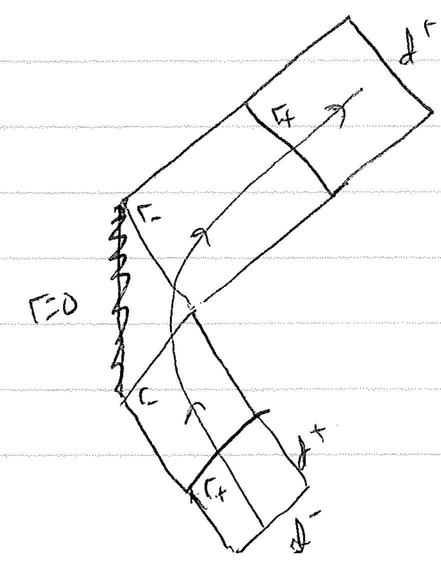
look at Penrose diagram for ingoing EF coord patch, infalling particle:



For outgoing particle, outgoing EF coords;



Both coord patches overlap + agree inside r_-, so full manifold that the particle sees is



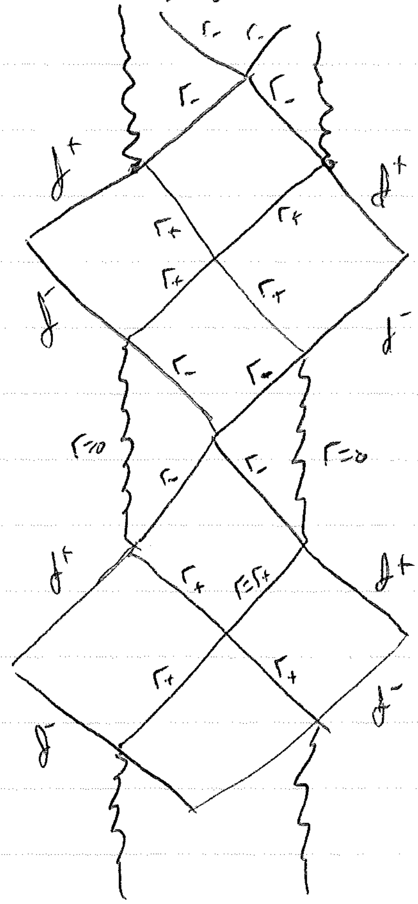
Two distinct asym. flat regions connected by tunnel through BH

2 copies of r_+ and r_-

This is freefalling particle.

- Gravity is repulsive inside $r = r_-$
  (i.e. $m_{eff} = m - \frac{Q^2}{2r}$ becomes negative)

- Actually, RN has ∞ number of tunnels. Full penrose diagram:



- After exiting white hole, observer can turn on rocket and go down next BH

- or, for $E<1$, observer is bound between $r_{min} + r_{max}$, passes thru tunnel each time.

- Astrophysically, do not expect charge on BHs
- BH tunnels unstable to perturbations
- r_- is "Cauchy surface" : anything inside r_- is influenced by singularity.