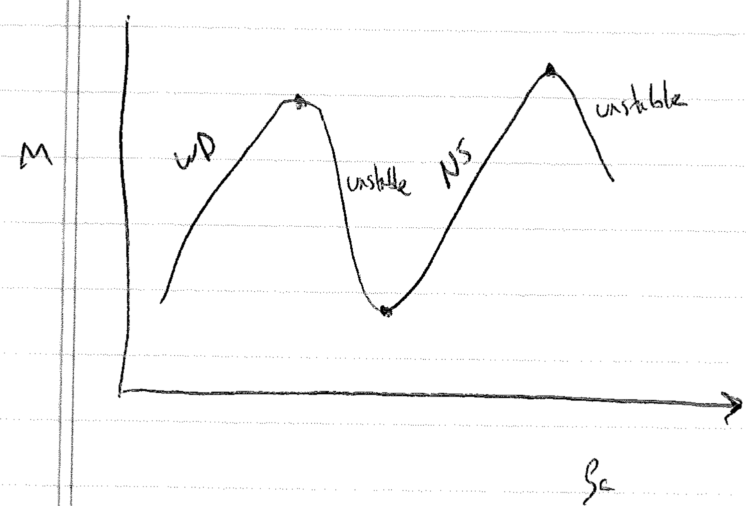
Stability of stellar models

Last time: why is ω_n^2 real?

IF pert. eqs. invariant under $t \to -t$, then only ω_n^2 appears when assuming perturbation $\sim e^{-i\omega_n t}$

$\Rightarrow$ no $i\omega_n$ terms

$$\left[\text{wave eqn} \quad \underbrace{\frac{d^2\psi}{dt^2}}_{-\omega^2} - \frac{d^2\psi}{dx^2} + \xi \underbrace{\frac{d\psi}{dt}}_{\substack{\text{damping term} \\ \to -i\omega}} = 0 \right]$$



Instability caused by properties of EOS, not necessarily by GR.
Depends crucially on EOS

Return to black holes

Last term: spherical b.h. Any spherically symmetric spacetime

Schwarzschild $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$

"Problem" at $r = 2M$: $r < 2M$ not visible to observer outside $r = 2M$

"Event horizon"

- infinite redshift between $r = 2M$ and $r = \infty$
- infinite acceleration for observer hovering at $r = 2M + \varepsilon$

- Particle can fall thru $r = 2M$, but Observer never sees particle pass $r = 2M$ b/c $t \to \infty$ at $r = 2M$
- Curvature ($R_{\alpha\beta\gamma\delta}$) finite at $r = 2M$

$\Rightarrow$ Coordinate singularity (t = bad coord)

"Problem" at $r = 0$: $R_{\alpha\beta\gamma\delta} \to \infty$

Physical singularity.

Kruskal-Szekeres coords:

- let $r^* = r + 2M \ln\left(r/2M - 1\right)$

"tortoise coord"

- let $\tilde{u} = t - r^*$ "Eddington-Finkelstein coords" outgoing
  $\tilde{v} = t + r^*$ ingoing

- let $\bar{u} = -e^{-\tilde{u}/4M}$
  $\bar{v} = e^{+\tilde{v}/4M}$

- let $\bar{u} = v - u$
  $\bar{v} = v + u$ Kruskal-Szerkes coords

Schwartschild $\Rightarrow$ $$\boxed{ds^2 = \frac{32M^3}{r}\left(-dv^2 + du^2\right)e^{-r/2M} + r^2 d\Omega^2 \qquad r = r(v,u)}$$

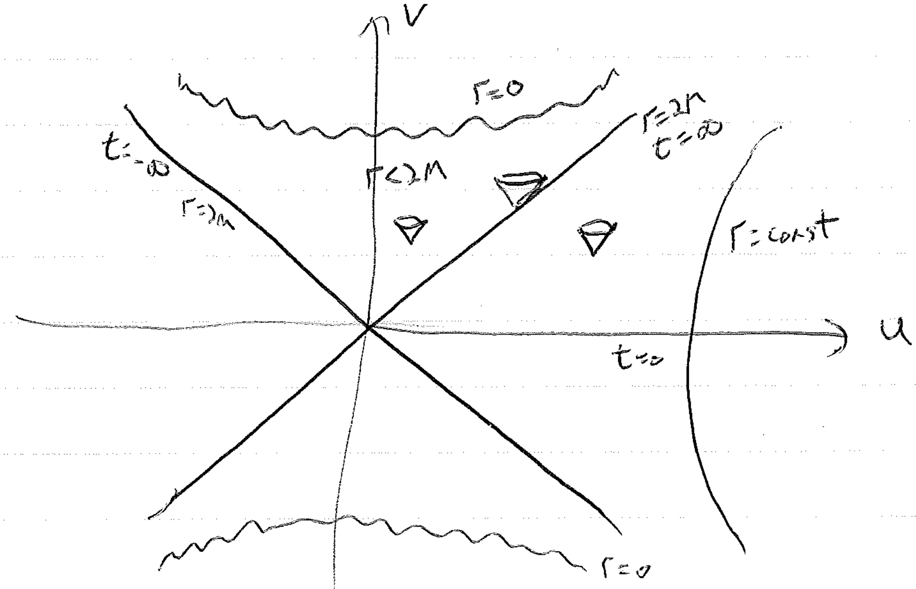
$-\bar{u}\bar{v} = \left(\frac{r}{2M} - 1\right)e^{r/2M}$

$u^2 - v^2 =$

$r = 0 \Leftrightarrow u^2 - v^2 = -1$

$r = 2M \Leftrightarrow u^2 - v^2 = 0$

$r < 2M$, g_{00} + g_{rr} flip sign



Penrose – Carter diagram

Goal: see the entire spacetime in one diagram

Example: Minkowski: $ds^2 = -dt^2 + dr^2 \quad + r^2 d\Omega^2$

Choose coords $\quad t + r = \tan \frac{1}{2}(\xi + \psi)$

$$t - r = \tan \frac{1}{2}(\psi - \xi)$$

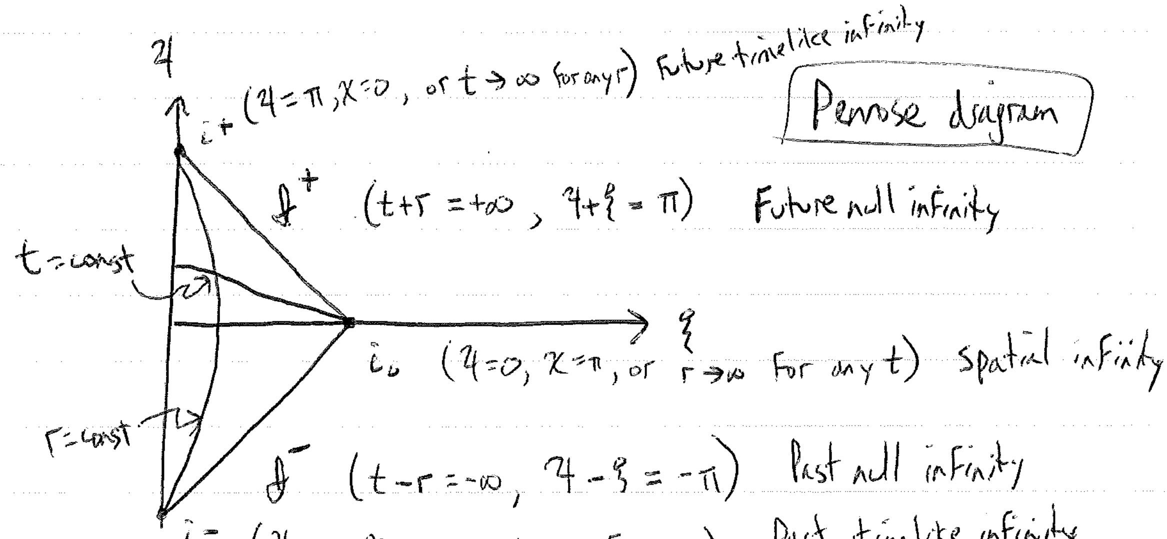
so $\psi + \xi \to \pm\pi$ as $t + r \to \pm\infty$

$\psi - \xi \to \pm\pi$ as $t - r \to \pm\infty$

$$dt + dr = \frac{1}{2}\frac{d\xi + d\psi}{\cos^2 \frac{1}{2}(\xi+\psi)} \qquad dt - dr = \frac{1}{2}\frac{-d\xi + d\psi}{\cos^2 \frac{1}{2}(\xi - \psi)}$$

$$\Rightarrow -dt^2 + dr^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2\left(\frac{1}{2}(\xi+\psi)\right)\cos^2\left(\frac{1}{2}(\xi-\psi)\right)} = \Omega^2(\xi,\psi)\left(-d\psi^2 + d\xi^2\right)$$

$\boxed{\text{Penrose diagram}}$

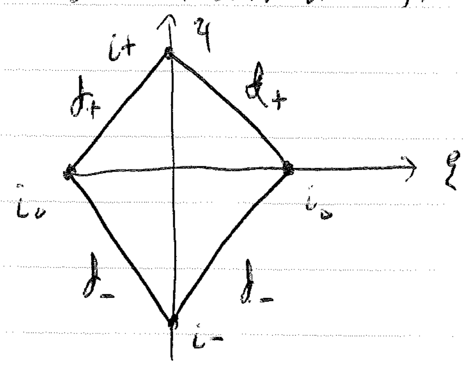


- All spacelike geodesics begin/end at i_0

- All timelike geodesics begin at i_-, end at i_+

- All null geodesics are at $\pm 45°$, begin at $\mathscr{I}_-$, end at $\mathscr{I}_+$

Penrose Diagram in t,x instead of t,r



$$t+x = \tan\tfrac{1}{2}(\xi+\eta)$$
$$t-x = \tan\tfrac{1}{2}(\eta-\xi)$$

$i_+, i_-, i_0, \mathscr{I}_+, \mathscr{I}_- \Rightarrow$ "conformal infinity"

"Asymptotically Flat" $\Rightarrow$ spacetime with the same conformal infinity as Minkowski

Penrose diagram for Schwartzschild:

Start with Kruskal coords: $ds^2 = \frac{32M^3}{r}\left(-dv^2+du^2\right)e^{-r/2m} + r^2 d\Omega^2$

Let $v + u = \tan \frac{1}{2}(\xi + \psi)$

$v - u = \tan \frac{1}{2}(\psi - \xi)$

$$\Rightarrow ds^2 = \frac{32M^3}{r}\left(-d\psi^2 + d\xi^2\right)e^{-r/2m}\frac{1}{4\cos^2\left(\frac{\psi+\xi}{2}\right)\cos^2\left(\frac{\psi-\xi}{2}\right)} + r^2 d\Omega^2$$

Note $r=0$ when $u^2 - v^2 = -1$

$$\Rightarrow \tan\left(\frac{\xi+\psi}{2}\right) \cdot \tan\left(\frac{\psi-\xi}{2}\right) = +1$$

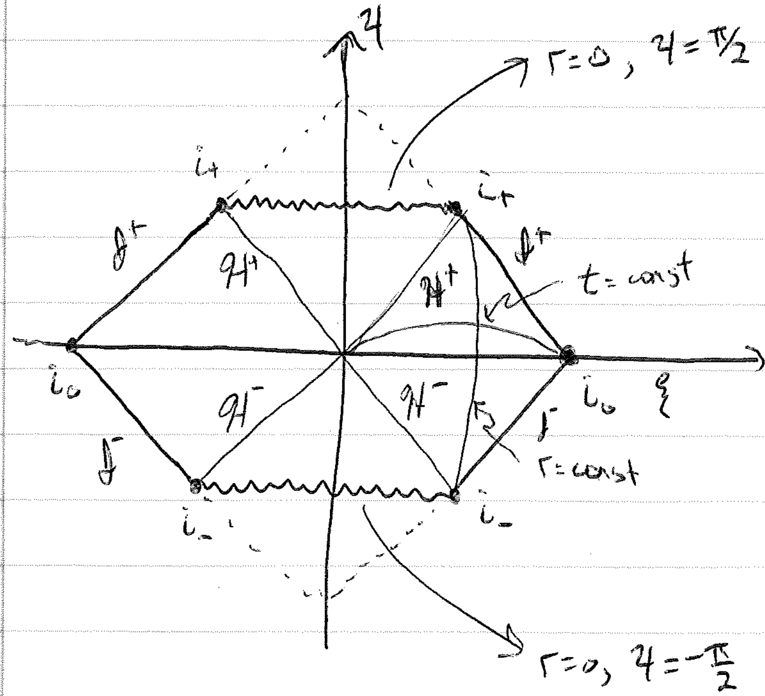
$$\text{or } \tan\left(\frac{\psi-\xi}{2}\right) = \frac{1}{\tan\left(\frac{\xi+\psi}{2}\right)}$$

$$\Rightarrow \frac{1}{2}(\psi - \xi) = \frac{\pi}{2} - \frac{1}{2}(\psi + \xi)$$

$$\text{or } \frac{1}{2}(\psi - \xi) = -\pi/2 - \frac{1}{2}(\psi + \xi)$$

$\Rightarrow \psi = \pi/2$ or $\psi = -\pi/2$ for $r=0$

Horizons: $u^2 - v^2 = 0 \Rightarrow \xi = \pm\psi$

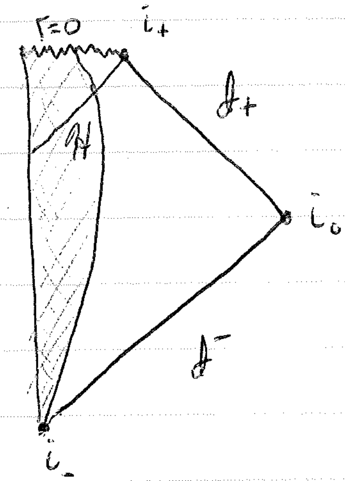


i_+ and i_- are distinct from $r=0$

Two asymptotically flat universes.

Light rays at $45°$

Penrose diagram of collapsing star

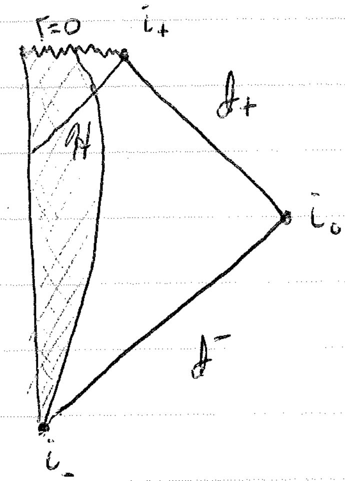


Interior of star is not Schwarzschild

No white hole.

No other universe.

Penrose diagram of collapsing star



Interior of star is not Schwarzschild

No white hole.

No other universe.

Reissner - Nordström black hole

- RN metric describes static, spherically symmetric black hole w/mass M + electric charge Q.

- Solves Einstein + Maxwell equations (vacuum otherwise)

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega^2$$

where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

$$A_\mu = \left(\frac{Q}{r}, 0, 0, 0\right)$$

Q = charge, measured by distant particles

- For $Q = 0$, $\Rightarrow$ Schwarzschild

- Can think of effective "mass" $m_{eff}(r) = M - \frac{Q^2}{2r}$ Then RN is like Schwarzschild

- <u>Singularities</u>: like Schwarzschild, true singularity at $r=0$

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \to 0 \quad \text{as} \quad r \to 0$$

- <u>Horizons</u> / coordinate singularities

Like Schwarzschild, coords go bad when $f(r) = 0 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

<u>Two</u> roots:

$$r_+ = M + \sqrt{M^2 - Q^2}$$
$$r_- = M - \sqrt{M^2 - Q^2}$$

Three cases

① $|Q| \leq M \Rightarrow r_+$ and r_- are real, $r_+ > r_-$

Event horizon at $r = r_+$ (like Schwarzschild)

Inner horizon $r = r_-$ is inside the BH. (more later)

② $|Q| = M \Rightarrow r_+ = r_- = M$ <u>Extremal</u> RN black hole

③ $|Q| > M \Rightarrow$ No real solns for r_+, r_-

- $f(r)$ always $\neq 0$ for all real r.
- No horizons, no coord. singularity
- <u>Not</u> a black hole.
- Still have $r=0$ physical singularity, "naked"

For all the world to see.
Not "clothed" by a horizon.

- Null radial geodesics $ds^2 = 0 = -f(r)dt^2 + \frac{1}{f(r)}dr^2$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = (f(r))^2 \qquad \frac{dr}{dt} = \pm |f(r)|$$

Never changes sign

- $|Q| > M$ Thought to be unphysical

Cosmic censureship conjecture (Penrose): There are NO naked singularities

From now on, we will assume $|Q| \leq M$.