Last time:

Metric for spherical star: $ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2$

Exterior: Schwartzschild: $\boxed{e^{-2\lambda} = 1 - \frac{2M}{r}}$

$\boxed{e^{2\Phi} = 1 - \frac{2M}{r}}$ (Sch 2)

M measured by particle orbits

Interior: $G_{\hat{0}\hat{0}} = 8\pi T_{\hat{0}\hat{0}} \Rightarrow e^{-2\lambda} = 1 - \frac{2m(r)}{r}$

where $\frac{dm}{dr} = 4\pi\rho r^2 \quad \left(\text{or } m(r) = \int_0^r 4\pi\rho r'^2 dr' \right)$

"mass inside radius r"

and $G_{\hat{r}\hat{r}} = 8\pi T_{\hat{r}\hat{r}} \Rightarrow \boxed{\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r(r-2m)}}$ (Int r)

and Euler $(T^{\alpha\beta}{}_{;\beta} = 0) \Rightarrow \boxed{(P+\rho)\frac{d\Phi}{dr} = -\frac{dP}{dr}}$

Match interior + exterior at surface of star:

$m(R) = \int_0^R 4\pi\rho r^2 dr = M \qquad R = \text{radius of star}$

M looks like volume integral of ρ but is not!

3d volume element is $dV = e^{\lambda} r^2 \sin^2\theta \, dr\, d\phi\, d\theta$ $\left(\begin{matrix}\text{not} \\ r^2 \sin^2\theta \, dr\, d\theta\, d\phi\end{matrix}\right)$

Why? 4d volume element is $\tilde{\varepsilon} = \underbrace{\varepsilon_{0123}}_{\sqrt{-g}} \tilde{d}x^0 \wedge \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$

$$= \sqrt{-g}\, dt \wedge dr \wedge d\theta \wedge d\phi \qquad \text{4-form}$$

3d volume element is

$$^{(3)}\tilde{\varepsilon} = \varepsilon_{123} \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$$

$$= \varepsilon_{123}\, dr \wedge d\theta \wedge d\phi$$

but $\varepsilon_{\alpha\beta\gamma} = n^{\delta} \varepsilon_{\delta\alpha\beta\gamma}$ $\qquad n^{\delta}$ = normal to 3-surface

$n^{\delta} = (n^0, 0, 0, 0)$

$\vec{n} \cdot \vec{n} = -1 \Rightarrow n^0 = e^{-\Phi}$

So $\varepsilon_{123} = \sqrt{-g}\, e^{-\Phi}$

$$= e^{\Phi} e^{\lambda} r^2 \sin^2\theta\, e^{-\Phi}$$

$$= e^{\lambda} r^2 \sin^2\theta$$

So for instance, total # of baryons in star

$$N = \int n\, dV = \int n e^{\lambda} r^2 \sin^2\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^R 4\pi r^2 e^{\lambda} n\, dr \qquad \left(\text{not } \int_0^R 4\pi r^2 n\, dr\right)$$

rest mass per baryon

Total rest mass in star $= M_0 = \mu_0 N = \int_0^R 4\pi r^2 e^{\lambda} \mu_0 n\, dr$

$\nearrow$ energy density $\nearrow$ rest mass density

Internal energy density $= \rho - \mu_0 n$ All local energy that's not rest mass

so internal energy is $U = \int (\rho - \mu_0 n)\, dV = \int_0^R 4\pi r^2 e^{+\lambda} (\rho - \mu_0 n)\, dr$

Note that $M - M_0 - U$ is not zero:

$$M - M_0 - U = \int_0^R 4\pi r^2 \rho \left(1 - e^{+\lambda}\right) dr$$

$=$ gravitational binding energy Ω

i.e. $M = M_0 + U + \Omega$

also $1 - e^{\lambda} = 1 - \left(1 - \frac{2m(r)}{r}\right)^{-1/2} < 0$

Binding energy is negative

Newtonian Limit $\Omega = \int_0^R 4\pi r^2 \rho \left(1 - e^{\lambda}\right) dr$

$$= \int_0^R 4\pi r^2 \rho \left(1 - \left(1 - \frac{2m}{r}\right)^{-1/2}\right) dr$$

$$= -\int_0^R 4\pi \rho \frac{m}{r} r^2\, dr$$

Newtonian potential energy

How to build a star

1. Specify equation of state $P = P(n)$ or $\rho = \rho(n)$ or $P = P(\rho)$ (microphysics)

2. Specify central value of n, P, or ρ.

3. Integrate system of ODEs
$$\begin{cases} \dfrac{dm}{dr} = 4\pi r^2 \rho \\[2ex] \dfrac{dP}{dr} = -\dfrac{(P+\rho)(4\pi r^3 P + m)}{r(r-2m)} \end{cases}$$

From $r=0$ with initial conditions $m(0)=0$, $P(0) = P_c$

(use EOS to go between $P \Leftrightarrow \rho$)

Integrate until pressure is zero, call that radius R
$$P(R) = 0$$
$$m(R) = M$$

Now you have $m(r)$, $P(r)$ [and $\rho(r)$, $n(r)$ from EOS]

4. Integrate $\dfrac{d\Phi}{dr} = \dfrac{m + 4\pi r^3 P}{r(r-2m)}$ to get $\Phi(r)$

Choose integration const so that $e^{2\Phi(R)} = 1 - \dfrac{2M}{R}$

to match interior metric to exterior Schwarzschild sol'n.

Special case where eqs are analytic:

$\rho = \text{const}$

Then $m(r) = \frac{4}{3}\pi r^3 \rho \qquad M = \frac{4}{3}\pi R^3 \rho$

TOV : $$\frac{dP}{dr} = \frac{-(P+\rho)\left(\frac{4\pi}{3} r^3 \rho + 4\pi r^3 P\right)}{r\left(r - 8\pi r^3 \rho/3\right)}$$

$$\Rightarrow \frac{dP}{4\pi(P+\rho)(\rho/3+P)} = \frac{-r\,dr}{1-8\pi r^2 \rho/3}$$

$$\Rightarrow \left(\frac{dP}{\rho/3+P} - \frac{dP}{\rho+P}\right)\frac{1}{8\pi\rho/3} = \frac{-r\,dr}{1-8\pi\rho/3\, r^2}$$

$$\Rightarrow \frac{1}{8\pi\rho/3} \ln\left[\frac{\rho/3+P}{(\rho+P)}\right] = \frac{1}{16\pi\rho/3} \ln\left(1 - \frac{8\pi r^2 \rho}{3}\right) + \text{const}$$

$$\Rightarrow \frac{P+\rho/3}{P+\rho} = C\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}$$

$P = 0$ at $R = r$

$$\Rightarrow \frac{1}{3} = C\left(1 - \frac{2M}{R}\right)^{1/2}$$

$$\Rightarrow \quad \boxed{\frac{P}{\rho} = \frac{\left(1-\frac{2Mr^2}{R^3}\right)^{1/2} - \left(1-\frac{2M}{R}\right)^{1/2}}{3\left(1-\frac{2M}{R}\right)^{1/2} - \left(1-\frac{2Mr^2}{R^3}\right)^{1/2}}} \qquad r<R \, .$$

To get Φ:

$$\frac{d\Phi}{dP} = \frac{d\phi/dr}{dP/dr} = \frac{-1}{\rho+P} \quad \Rightarrow \quad \Phi = -\ln(\rho+P) + \text{const}$$

$$\Rightarrow \quad e^{\Phi} = \frac{\text{const}}{1+P/\rho}$$

Match at $r=R$
to Schwarzschild

$$e^{\Phi} = \text{const} = \left(1-\frac{2M}{R}\right)^{1/2}$$

$$\Rightarrow \quad e^{\Phi} = \frac{\left(1-\frac{2M}{R}\right)^{1/2}}{1+P/\rho} \qquad r<R$$

$$= \frac{3}{2}\left(1-\frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2Mr^2}{R^3}\right)^{1/2} \qquad r<R$$

(Newtonian limit: $M/R \ll 1$

$$e^{\Phi} \sim 1 - \frac{1}{2}\frac{M}{R}\left(3-\frac{r^2}{R^2}\right)$$

$$e^{\Phi} \sim 1+\Phi \qquad \Phi \sim -\frac{1}{2}\frac{M}{R}\left(3-r^2/R^2\right)$$

Newtonian result
for Φ = Newtonian potential

For $\rho = $ const solution

Central pressure $\quad P$ at $r=0 = P_c = \rho \dfrac{1-(1-2M/R)^{1/2}}{3(1-2M/R)^{1/2}-1}$

Notice central pressure $\to \infty$ as $\dfrac{2M}{R} \to \dfrac{8}{9}$

Cannot have uniform density star w/ $\dfrac{2M}{R} > \dfrac{8}{9}$

(Turns out that $\dfrac{2M}{R} < 8/9$ for <u>any</u> $\rho(r)$ with $\dfrac{d\rho}{dr} < 0$)

Notice:

Proper distance in radial direction:

$$dr_{proper} = e^{\lambda} dr \qquad \left(\sqrt{ds^2} \text{ with } t, \theta, \phi = \text{const} \right)$$

$$\text{So} \quad \frac{dr_{proper}}{dr} = e^{\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1/2} > 1$$

but area of sphere about origin is $4\pi r^2$

circumference of circle is $2\pi r$

$$\text{So} \quad \frac{C}{2\pi r_{proper}} < 1$$

$$\frac{A}{4\pi r^2_{proper}} < 1$$

Embedding Diagram For const density star?

Inside $\quad m(r) = \frac{4\pi r^3}{3}\rho \qquad ds^2 = \left(1-\frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\phi^2$

$$\frac{dz}{dr} = \frac{1}{\left(r/2m(r) - 1\right)^{1/2}} = \frac{1}{\left(\frac{3}{8\pi r^2\rho} - 1\right)^{1/2}}$$

$$z = \int_0^r \frac{1}{\left(\frac{3}{8\pi\rho} - r'^2\right)^{1/2}} r' dr' = \left(\frac{3}{8\pi\rho}\right)^{1/2} - \left(\frac{3}{8\pi\rho} - r^2\right)^{1/2}$$

let $a = \left(\frac{3}{8\pi\rho}\right)^{1/2} = R\left(\frac{R}{2m}\right)^{1/2}$ $\qquad$ so $z(r) = a - (a^2 - r^2)^{1/2}$

$$(z-a)^2 + r^2 = a^2$$

Sphere of radius a centered at $z = a$