Physics 236b - General Relativity

Topics

www.tapir.caltech.edu/~scheel/P236

- Spherical Stars
- Black Holes
- Gravitational Waves
- Cosmology
- Numerical Relativity / 3+1 formulation

Fluids in GR

Fluid: Approximation of matter — At each point in spacetime, Fluid has unique 4-velocity $\vec{u}$

— No anisotropy

Fluid parameters: $\vec{u}$ = 4-velocity of Fluid

ρ = energy density in rest Frame of Fluid

p = pressure " "

n = baryon density " "

s = entropy per baryon " "

T = temperature " "

Fluid laws:

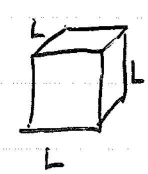
① Baryon number conservation: $\nabla \cdot (n\vec{u}) = 0$

$$(nu^\alpha)_{;\alpha} = 0$$

Why? First, show $\vec{\nabla} \cdot \vec{u} = \frac{1}{V}\frac{dV}{d\tau}$ where V is the proper volume occupied by N baryons.

Go into rest Frame (LLF)

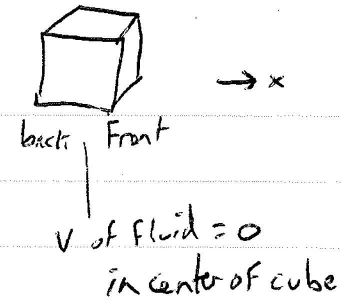
⇒ let volume element be a cube



Δx, Δy, Δz of cube is L, will change in time δt

$V = $ 3-velocity of Fluid

$$\delta(\Delta x) = V^x\Big|_{\text{Front}} \delta t - V^x\Big|_{\text{back}} \delta t$$



$$= V^x{}_{,x} \, L \, \delta t$$

similarly for $\delta(\Delta y)$, $\delta(\Delta z)$

$$\text{so} \quad \delta(\Delta x \Delta y \Delta z) = L^3 V^i{}_{,i} \, \delta t$$

$$\Rightarrow \quad \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = V^i{}_{,i} \quad \text{in LLF}$$

$$\Rightarrow \quad \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{d\tau} = u^\alpha{}_{;\alpha} \quad \text{in LLF} \qquad \begin{pmatrix} u^0 = 1 \text{ in LLF} \\ u^i = V^i \text{ in LLF} \\ t = \tau \text{ in LLF} \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{d\tau} = u^\alpha{}_{;\alpha} \quad \text{in all frames}$$

Now go back to show baryon conservation $\Rightarrow (n u^\alpha)_{;\alpha} = 0$

Proper volume $\mathcal{V}$ occupied by N baryons changes over time, but N does not

$$N = n\mathcal{V} \quad \text{so} \quad \frac{d}{d\tau}(n\mathcal{V}) = 0$$

$$0 = \frac{1}{\mathcal{V}} \frac{d}{d\tau}(n\mathcal{V}) = \frac{n}{\mathcal{V}} \frac{d}{d\tau}\mathcal{V} + \frac{\mathcal{V}}{\mathcal{V}} \frac{d}{d\tau} n$$

$$= n u^\alpha{}_{;\alpha} + u^\alpha n_{;\alpha} \qquad \frac{d}{d\tau} = \nabla_{\vec{u}}$$

$$= (n u^\alpha)_{;\alpha}$$

Next Fluid law:

2nd Law of Thermodynamics:

$$\frac{ds}{d\tau} \geq 0$$

assuming no heat flow

$\boxed{\text{Special case: Local Thermodynamic Equilibrium } \frac{ds}{d\tau} = 0}$

1st Law of Thermo:

$$d\begin{pmatrix}\text{energy in volume} \\ \text{containing } N \\ \text{baryons}\end{pmatrix} = -P\,dV + T\,dS$$

$$\Rightarrow \quad d\left(\rho N/n\right) = -P\,d\left(\frac{N}{n}\right) + T\,dS \quad \text{— capital S}$$

$$\Rightarrow \quad d\left(\rho/n\right) = -P\,d\left(\frac{1}{n}\right) + T\,ds \quad \text{— small s}$$

$$\Rightarrow \quad \boxed{d\rho = \frac{P+\rho}{n}\,dn + nT\,ds}$$

what is "d"? exterior deriv or directional deriv along any vector field $\vec{v}$

$$\tilde{d}\rho = \frac{P+\rho}{n}\,\tilde{d}n + nT\,\tilde{d}s$$

$$\nabla_{\vec{v}}\,\rho = \frac{P+\rho}{n}\,\nabla_{\vec{v}}\,n + nT\,\nabla_{\vec{v}}\,s$$

Special case $\vec{v} = \vec{u}$

$$\boxed{\frac{d\rho}{d\tau} = \frac{P+\rho}{n}\,\frac{dn}{d\tau} + nT\,\frac{ds}{d\tau}}$$

Fluid description incomplete without

<u>Equation of State</u>: Some Function

$\rho(n,s)$ depends on microphysics

(Note: given $\rho(n,s)$, can compute $P(n,s) = n\left(\frac{\partial\rho}{\partial n}\right)\Big|_s - \rho$

$T(n,s) = \frac{1}{n}\left(\frac{\partial\rho}{\partial s}\right)_n$)

From 1st law

Special case: <u>Perfect Fluid</u> $\frac{ds}{d\tau} = 0$ $s =$ constant along flow

adiabatic

No shear stress
No viscosity
No heat conduction

Then equation of state is $\rho(n)$ [or $P(\rho)$ or $n(\rho)$]

For perfect Fluid, $T^{\alpha\beta} = (P+\rho)U^\alpha U^\beta + P g^{\alpha\beta}$

Examples: White Dwarf + Neutron Star
Held up by degeneracy pressure + strong nuclear force
Thermal contribution to pressure is negligible
$\Rightarrow$ can treat $T \to 0$ and $s \to 0$ everywhere
not only adiabatic, but isentropic

For perfect fluid, $T^{\alpha\beta} = (P+\rho)u^\alpha u^\beta + P g^{\alpha\beta}$

$T^{\alpha\beta}{}_{;\beta} = 0$ gives 2 equations $\begin{pmatrix} \alpha = 0 \\ \alpha = i \end{pmatrix}$

① $\Rightarrow \dfrac{d\rho}{d\tau} = -(\rho+P)u^\alpha{}_{;\alpha} = (\rho+P)\dfrac{dn}{d\tau}$ 1st law ($ds=0$)

② $\Rightarrow (P+\rho)\nabla_{\vec{u}}\vec{u} = -\nabla P - \vec{u}\,\nabla_{\vec{u}} P$

Same as $(P+\rho)\dfrac{d\vec{u}}{d\tau} = -\nabla P - \vec{u}\,\dfrac{dP}{d\tau}$

Same as $(P+\rho)u_{\alpha;\beta}u^\beta = -P_{;\alpha} - u_\alpha u^\beta P_{;\beta}$

Euler equation
"F = ma"

Spherical Stars

Recall in derivation of Schwarzschild, before we plugged in $T_{\mu\nu}=0$, we found most general spherically symmetric metric was

$$\boxed{ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2 d\Omega^2}$$

$$\left(d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

"Schwarzschild coordinates"

Coordinate r has geometrical meaning:

$$r = \left(\frac{\text{area of sphere centered on origin}}{4\pi}\right)^{1/2} = \frac{\text{circumference of circle}}{2\pi}$$

Coordinates θ, ϕ are usual spherical coords $\quad 0 \le \theta \le \pi$

$$0 \le \phi < 2\pi$$

- We want to solve for a spherical star

Strategy:
- Outside star, $T_{\mu\nu}=0$, $\Rightarrow$ Schwarzschild
- Inside star, still need to find metric
- Match interior/exterior solutions at surface of star.

Interior solution:

Assume perfect Fluid : $T_{\alpha\beta} = (P+\rho)\, u_\alpha u_\beta + P g_{\alpha\beta}$

Assume static : $\frac{\partial}{\partial t}$ is a Killing vector

$$\Rightarrow \quad \Phi(r,t) = \Phi(r)$$
$$\lambda(r,t) = \lambda(r)$$

(Note For vaccum we did not need to assume $\frac{\partial}{\partial t} = $ K.V.

We proved it. Here, $\frac{\partial}{\partial t}$ would not be a K.V. if matter is moving)

<u>Static</u> $\Rightarrow$ $u^i = \frac{dx^i_{\text{Fluid}}}{d\tau} = 0$ (also $u_i = g_{i\alpha} u^\alpha = 0$)

$$\vec{u}\cdot\vec{u} = -1 \quad \Rightarrow \quad u^0 = e^{-\Phi}$$

$$\Rightarrow \quad T^{00} = \rho e^{-2\Phi} \quad T^{rr} = P e^{-2\lambda} \quad T^{\theta\theta} = P/r^2 \quad T^{\phi\phi} = \frac{P}{r^2 \sin^2\theta}$$

5 unknowns: $P, \rho, n, \Phi, \lambda$

- Equation of state specifies $P(n)$ + $\rho(n)$
- Need 3 equations for 3 more unknowns

$T^{\mu\nu}{}_{;\nu} = 0 \quad \Rightarrow$ 1st law (trivial for static situations)
Euler

Euler is $(P+\rho)\, u_{\alpha;\beta} u^\beta = -P_{,\alpha} - u_\alpha u^\beta P_{,\beta}$

only nontrivial piece is $\alpha = r$ (recall $u_r = 0$) static

$$(P+\rho)\, u_{r;\beta} u^\beta = -P_{,r}$$

$$\Rightarrow \quad -P_{,r} = (P+\rho)\, u^\beta (\cancelto{0}{u_{r,\beta}} - \Gamma^\alpha{}_{r\beta} u_\alpha)$$

$$= -(P+\rho)\, \Gamma^t{}_{rt} \underbrace{u_t u^t}_{-1}$$

$$= (P+\rho)\, \Gamma^t{}_{rt}$$

Can plug in to metric to get $\Gamma^t{}_{rt} = \frac{1}{2} g^{tt}(g_{tt,r}) = \Phi_{,r}$

$$\Rightarrow \quad \boxed{(P+\rho)\Phi_{,r} = -P_{,r}} \quad ①$$

(Newtonian limit: Φ = gravitational potential

$P \ll \rho \quad \Rightarrow \quad \underbrace{P_{,r}}_{\text{Pressure force}} = -\underbrace{\rho\Phi_{,r}}_{\text{gravitational force}}$)

Now we need to use $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

Easiest in orthonormal frame
$$\tilde{\omega}^{\hat{0}} = e^{\Phi} dt$$
$$\tilde{\omega}^{\hat{r}} = e^{\lambda} dr$$
$$\tilde{\omega}^{\hat{\theta}} = r\, d\theta$$
$$\tilde{\omega}^{\hat{\phi}} = r \sin\theta\, d\phi$$

Then $T_{\hat{0}\hat{0}} = \rho$, $T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = P$, others zero

You get $G_{\hat{0}\hat{0}} = \frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\lambda}\right)\right] = 8\pi\rho$

So <u>define</u> $\boxed{m(r) \equiv \frac{1}{2} r\left(1-e^{-2\lambda}\right)}$ (or $e^{-2\lambda} = 1 - \frac{2m(r)}{r}$)

Then $\boxed{\frac{dm}{dr} = 4\pi\rho r^2}$ ② (or $m(r) = m(0) + \int_0^r 4\pi r^2 \rho\, dr$)

m(0) must be zero so metric is regular at $r=0$

We did $G_{\hat{0}\hat{0}}$, now do $G_{\hat{r}\hat{r}} = 8\pi T_{\hat{r}\hat{r}}$

$$G_{\hat{r}\hat{r}} = -(1-e^{-2\lambda})/r^2 + \frac{2e^{-2\lambda}}{r}\Phi_{,r}$$

$$= 8\pi P$$

$$\Rightarrow -\frac{2m}{r^3} + 2\left(1-\frac{2m}{r}\right)\frac{\Phi_{,r}}{r} = 8\pi P$$

$$\Rightarrow \boxed{\frac{d\Phi}{dr} = \frac{m+4\pi r^3 P}{r(r-2m)}} \quad ③$$

(Newtonian limit $\frac{d\Phi}{dr} = \frac{m}{r^2}$)

$$①+③ \Rightarrow \boxed{\frac{dP}{dr} = \frac{-(\rho+P)(m+4\pi r^3 P)}{r(r-2m)}}$$

Tolman-Oppenheimer-Volkoff (TOV) eqn. (Newtonian limit: $\frac{dP}{dr} = -\rho\frac{m}{r^2}$)

Note: $\left.\frac{dP}{dr}\right|_{GR} > \left.\frac{dP}{dr}\right|_{Newtonian}$

"Pressure regeneration"

Pressure inside a star is larger for GR than Newtonian