Solutions Ph 236b – Week 4

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Problem 1

Part (a)

The redshift factor between an observer at radius r in the Schwarzchild metric to an observer at infinity is given by

$$\frac{dt}{d\tau} = \frac{1}{1 - 2M/r} \tag{1.1}$$

An observer measures the time dilation of Miller's planet to be

$$\frac{dt}{d\tau} = \frac{7\text{years}}{1\text{hour}} = 7 * 365 * 24 = 61320 \tag{1.2}$$

Solving for the radius gives $r = (2 + \epsilon)M$ for a very small $\epsilon \sim 10^{-10}$. Since this is much smaller than 1, $r < 3M$ which is the smallest possible circular orbit in Schwarzchild. So the black hole cannot be in a circular orbit in Schwarzschild.

Part (b)

The expression for the time coordinate of an orbiting body in in the Kerr metric is

$$\frac{dt}{d\tau} = \frac{-2Mra\tilde{L} + \tilde{E}[(r^2 + a^2)\Sigma + 2Mra^2 \sin^2\theta]}{\Delta\Sigma}. \tag{1.3}$$

For circular, corotating orbits,

$$\begin{aligned}
\tilde{E} &= \frac{r^{3/2} - 2Mr^{1/2} + aM^{1/2}}{r^{3/4}(r^{3/2} - 3Mr^{1/2} + 2aM^{1/2})^{1/2}} \\
\tilde{L} &= \frac{M^{1/2}(r^2 - 2aM^{1/2}r^{1/2} + a^2)}{r^{3/4}(r^{3/2} - 3Mr^{1/2} + 2aM^{1/2})^{1/2}}
\end{aligned} \tag{1.4}$$

Plug these into the expression for $\frac{dt}{d\tau}$. Use the approximations for the radius of the ISCO, $r \approx M(1 + 4^{1/3}\epsilon)$ corresponding to a spin of $a \approx M(1 - \epsilon^3)$. Expand the expression in powers of ϵ and solve for its value. See the mathematica notebook attached for an example of this. The resulting value for the spin is roughly $a \approx M(1 - 1.33564 \times 10^{-14})$.

Part (c)

To show that the basis one-forms given match the Kerr metric, simply compute

$$\begin{aligned} ds^2 =& \eta_{\hat{\mu}\hat{\nu}}\tilde{\omega}^{\hat{\mu}}\tilde{\omega}^{\hat{\nu}} \\ =& -(\tilde{\omega}^{\hat{0}})^2 + (\tilde{\omega}^{\hat{1}})^2 + (\tilde{\omega}^{\hat{2}})^2 + (\tilde{\omega}^{\hat{3}})^2 \\ =& -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)d\phi - adt\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned} \tag{1.5}$$

Part (d)

Both of the basis vectors and one-forms can be represented by a metric which transforms from the coordinate basis to the hatted basis, $\vec{e}_{\hat{\alpha}} = \Lambda^{\alpha}_{\hat{\alpha}}\vec{e}_{\alpha}$ and $\tilde{\omega}^{\hat{\beta}} = \lambda^{\hat{\beta}}_{\beta}\tilde{\omega}^{\beta}$. Here, finding the orthonormal vector basis corresponding to the given $\tilde{\omega}^{\hat{\beta}}$ is equivalent to inverting the matrix for the one-forms transposed so that $\Lambda^{\alpha T}_{\hat{\alpha}} = \lambda^{\hat{\alpha}-1}_{\alpha}$. Then the hatted basis vectors are given by

$$\vec{e}_{\hat{0}} = \frac{r^2+a^2}{\sqrt{\Sigma\Delta}}\vec{e}_0 + \frac{a}{\sqrt{\Sigma\Delta}}\vec{e}_3 \tag{1.6}$$

$$\vec{e}_{\hat{1}} = \sqrt{\frac{\Delta}{\Sigma}}\vec{e}_1 \tag{1.7}$$

$$\vec{e}_{\hat{2}} = \frac{1}{\sqrt{\Sigma}}\vec{e}_2 \tag{1.8}$$

$$\vec{e}_{\hat{3}} = -\frac{a\sin\theta}{\sqrt{\Sigma}}\vec{e}_0 - \frac{1}{\sin\theta\sqrt{\Sigma}}\vec{e}_3 \tag{1.9}$$

$$\tag{1.10}$$

Part (e)

To compute the vector $\vec{u}$, start by considering its one-form alternative, $\tilde{u} = -E\tilde{\omega}^0 + L\tilde{\omega}^3$. When changing basis, transforming components of a one-form is the same as transforming the basis vectors so $u_{\hat{\alpha}} = \Lambda^{\alpha}_{\hat{\alpha}}u_{\alpha}$. Thus, to get the components of the 4-velocity in the hatted basis, just use the transformation matrix found from the part above to perform the computation. After including the fact that for circular, equatorial orbits that $\sin\theta = 1$ and $\Sigma = r^2$,

$$\begin{aligned} u_{\hat{0}} =& u_0\Lambda^0_{\hat{0}} + u_3\Lambda^3_{\hat{0}} = -\tilde{E}\left(\frac{r^2+a^2}{r\sqrt{\Delta}}\right) + \tilde{L}\left(\frac{a}{r\sqrt{\Delta}}\right) \\ u_{\hat{3}} =& u_0\Lambda^0_{\hat{3}} + u_3\Lambda^3_{\hat{3}} = -\tilde{E}\left(\frac{-a}{r}\right) - \tilde{L}\left(\frac{1}{r}\right) \end{aligned} \tag{1.11}$$

To convert this one-form into a vector, simply multiply by the flat metric to get

$$\vec{u} = \frac{\tilde{E}(r^2 + a^2) - a\tilde{L}}{r\sqrt{\Delta}}\vec{e}_{\hat{0}} + \frac{a\tilde{E} - \tilde{L}}{r}\vec{e}_{\hat{3}} \tag{1.12}$$

Part (f)

Look in the attached script to see the actual computations. The orthonormality of $\vec{\lambda}_{\bar{2}} = \vec{e}_{\hat{2}}$ is straightforward because it is an orthonormal basis vector and the only $\vec{\lambda}$ which contains the 2-vector. Next up, $\vec{\lambda}_{\bar{0}} = \vec{u}$. Now, the expressions for $\tilde{E}$ and $\tilde{L}$ need to be substituted in, but doing so and computing the $\vec{u} \cdot \vec{u}$ does indeed give the expected value of -1 as any good 4-velocity should. So far, so good. To show orthonormality of $\vec{\lambda}_{\bar{1}}$ and $\vec{\lambda}_{\bar{3}}$, it is necessary to show that $A \cdot A = B \cdot B = 1$. This is in the attached notebook; running through the numbers for A and again making all of the necessary substitutions does give $A^2 = 1$. Similarly for $B^2 = 1$ but only when the correct B is used, obviously. After that, show that $\vec{u} \cdot B = 0$ and viola. Everything basically falls through from there.

Part (g)

Tidal forces in GR are computed using the geodesic deviation equation,

$$\frac{D^2 n^{\bar{\alpha}}}{D\tau^2} = R^{\bar{\alpha}}{}_{\bar{0}\bar{\gamma}\bar{0}} n^{\bar{\gamma}} \tag{1.13}$$

Stretching/squeezing occurs in every direction, as can be seen from all the components of the Riemann tensor. Since for tidal forces, stretching occurs along the radial direction out from the black hole, we only need to look at the radial direction for the planet. If you look at the other directions, you will find there is a negative sign corresponding to compression. In the $\lambda_{\bar{a}}$ basis, the direction corresponding to radial, which is only in the $\vec{e}_{\hat{1}}$ direction, which rotates between $\lambda_{\bar{1}}$ and $\lambda_{\bar{3}}$ determined by the parameter $\Psi(\tau)$. For simplicity, pick Ψ such that $\cos\Psi = 1$. Thus, the $\lambda_{\bar{1}}$ is entirely radial and no other $\lambda_{\bar{a}}$ contains any radial dependence. For $\vec{n}$, pick it strictly in the radial ($\lambda_{\bar{1}}$) direction with a length corresponding to the radius of the earth. (You can leave Ψ arbitrary if you wish, but then your $\vec{n}$ will have a corresponding Ψ dependence in both the $\lambda_{\bar{1}}$ and $\lambda_{\bar{3}}$ directions).

$$\frac{D^2 n^{\bar{1}}}{D\tau^2} = -R^{\bar{1}}{}_{\bar{0}\bar{1}\bar{0}} n^{\bar{1}} \tag{1.14}$$

From the previous parts of the problem, we see that the spin is large, so assume that $a \sim M$. Similarly, for a circular orbit for that high of a spin, we can assume that $r \sim M$. In this limit, it turns out that $\tilde{E} \approx 1/\sqrt{3}$ and $\tilde{L} \approx 2M/\sqrt{3}$, so $K \approx M^2/3$. Then the geodesic deviation equation becomes

$$\frac{D^2 n^{\bar{1}}}{D\tau^2} = -R_{\bar{1}\bar{0}\bar{1}\bar{0}} n^{\bar{1}} - R_{\bar{1}\bar{0}\bar{3}\bar{0}} n^{\bar{3}} \tag{1.15}$$

$$\approx (4\cos^2\Psi - 1)\frac{1}{M^2} n^{\bar{1}} + 4\sin\Psi\cos\Psi \frac{1}{M^2} n^{\bar{3}}, \tag{1.16}$$

$$\approx \frac{4\cos^2\Psi - 1 + 4\sin\Psi\cos\Psi}{M^2} r_{\text{earth}}, \tag{1.17}$$

$$\frac{D^2 n^{\bar{3}}}{D\tau^2} = -R_{\bar{3}\bar{0}\bar{3}\bar{0}} n^{\bar{3}} - R_{\bar{3}\bar{0}\bar{1}\bar{0}} n^{\bar{1}} \tag{1.18}$$

$$\approx (4\sin^2\Psi - 1)\frac{1}{M^2} n^{\bar{3}} + 4\sin\Psi\cos\Psi \frac{1}{M^2} n^{\bar{1}}, \tag{1.19}$$

$$\approx \frac{4\sin^2\Psi - 1 + 4\sin\Psi\cos\Psi}{M^2} r_{\text{earth}}, \tag{1.20}$$

$$\frac{D^2 n^{\bar{2}}}{D\tau^2} = -R_{\bar{2}\bar{0}\bar{2}\bar{0}} n^{\bar{2}} \tag{1.21}$$

$$= \frac{-2}{M^2} n^{\bar{2}}, \tag{1.22}$$

$$\approx \frac{-2}{M^2} r_{\text{earth}}. \tag{1.23}$$

The maximum tidal force occurs when $4\cos^2\Psi - 1 + 4\sin\Psi\cos\Psi$ is a maximum or when $4\sin^2\Psi - 1 + 4\sin\Psi\cos\Psi$ is a maximum. This occurs at $\Psi = \pm\pi/8, \pm 3\pi/8$, and for these angles the maximum tidal force is $\frac{3.83}{M^2} r_{\text{earth}}$. Substituting in all the dimensionful constants and the radius of the earth, setting the maximum tidal force to to 9.8m/s^2, and solving for M, gives a black hole mass on the order of $M_{BH} \approx 3 \times 10^8 M_\odot$.