Solutions Ph 236b – Week 3

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Problem 1

Part (a)

The momentum of the particle is

\[ p^\mu = \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi^\mu} = g^{\mu\nu} (\pi_\nu - eA_\nu) \]  \hfill (1.1)

\[ = \pi^\mu - eA^\mu \]  \hfill (1.2)

Part (b)

The equation of motion for a charged particle with the 4-vector \( p^\mu \) in an electromagnetic field is given by the Lorentz force equation

\[ p^\alpha p^\mu \equiv eF^{\mu\alpha}p_\alpha \]  \hfill (1.3)

Part (c)

We can verify the answer from Part (b) using Hamilton’s equations of motion. Eq. (3) gives

\[ \frac{d\pi^\mu}{d\lambda} = \frac{\partial H}{\partial p^\mu} = \frac{1}{2} p_\alpha p_\beta g^{\lambda\alpha} g^{\sigma\beta} g_{\lambda\sigma,\mu} + e g^{\alpha\beta} p_\alpha (A_\beta)_\mu \]  \hfill (1.4)

But \( g_{\lambda\sigma} = 2\Gamma_{(\lambda\sigma)\mu} \) and \( A_\beta,\mu = A_{\mu,\beta} + F_{\mu\beta} \), so we have

\[ \frac{d\pi^\mu}{d\lambda} = p_\alpha p_\beta g^{\lambda\alpha} g^{\sigma\beta} \Gamma_{(\lambda\sigma)\mu} + e g^{\alpha\beta} \left( F_{\mu(\beta} P_{\alpha)} + A_{\mu, (\beta} P_{\alpha)} \right) \]  \hfill (1.5)

\[ = \frac{dp_\mu}{d\lambda} + e \frac{dA_\mu}{d\lambda} \]  \hfill (1.6)

We note that \( A_{\mu, (\beta} P_{\alpha)} = \frac{dA_\mu}{d\lambda} \) and

\[ p^\alpha p_{\mu,\alpha} = p^\alpha p_{\mu,\alpha} - \Gamma^\lambda_{\mu\alpha} p_\lambda \]  \hfill (1.7)

So finally Eq (1.5) becomes

\[ p^\alpha p^\mu \equiv eF^{\mu\alpha} p_\alpha \]  \hfill (1.8)

which is the result of part (1b)

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Part (d)

We know that since \( \frac{\partial H}{\partial \phi} = \frac{\partial H}{\partial t} = 0 \), this implies
\[
\frac{d\pi_\phi}{d\lambda} = \frac{d\pi_t}{d\lambda} = 0 \quad (1.9)
\]
and so \( \pi_\phi \) and \( \pi_t \) are conserved. But
\[
\pi_t = p_t + eA_t, \quad A_t = -\frac{Q}{r} \quad (1.10)
\]
\[
= p_t - \frac{eQ}{r} = -E \quad (1.11)
\]
and
\[
\pi_\phi = p_\phi + eA_\phi, \quad A_\phi = 0 \quad (1.12)
\]
\[
= p_\phi = L \quad (1.13)
\]
so \( E \) and \( L \) are conserved.

Part (e)

We can derive Eq (6) in the problem set from the fact that the particle rest mass is conserved:
\[
-\mu^2 = g_{\alpha\beta}p^\alpha p^\beta \quad (1.14)
\]
\[
= \left( \frac{dr}{d\lambda} \right)^2 g_{rr} + (p_0)^2 g^{00} + \frac{(p_\phi)^2}{r^2} \quad (1.15)
\]
\[
= \left( \frac{dr}{d\lambda} \right)^2 \frac{1}{f(r)} \left( \frac{eQ}{r} - E \right)^2 \frac{1}{f(r)} + \frac{L^2}{r^2} \quad (1.16)
\]
where
\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad (1.17)
\]
and \( g_{rr} = -g^{00} = \frac{1}{f(r)} \). Rearranging will give you Eq (6).

Problem 2

Part (a)

Go to the local Lorentz frame comoving with the surface of the star so that \( \vec{u} = (1, 0), \vec{p} = (E, \vec{p}), \vec{n} = (0, \eta) \). Now \( |\vec{u}| = 1 \) and \( |\vec{p}| = E = -\vec{p} \cdot \vec{u} \). Then
\[
\cos \theta = \frac{\vec{n} \cdot \vec{p}}{|\vec{n}||\vec{p}|} = \frac{-\vec{n} \cdot \vec{p}}{|\vec{n}||\vec{p}|} \quad (2.1)
\]
Since this final quantity is a scalar, it can be computed in any frame.
Part (b)

Without loss of generality, assume $\theta = \pi/2$, $p_\theta = 0$. For a photon,

$$\vec{p} \cdot \vec{p} = -\frac{1}{\alpha} (p_0)^2 + \alpha (p_r)^2 + \frac{1}{r} (p_\phi)^2$$

(2.2)

where $\alpha = 1 - 2M/r$. Since $p_0 = -E$ and $p_\phi = L$ and labeling $b = L/E$, then

$$p_r = \frac{E}{\alpha} \left( 1 - \frac{b^2}{r^2} \right)^{1/2}$$

(2.3)

Now let $v_s = \frac{dr}{dt} = \frac{u_r}{u_0}$. Then

$$\vec{u} \cdot \vec{n} = 0 = -u_0 n^0 \alpha + n^r u^r/\alpha \Rightarrow n^0 = n^r v_s/\alpha^2.$$

Now compute the other relevant quantities

$$\vec{p} \cdot \vec{n} = -En^0 + p_r n^r = \frac{u^r E}{\alpha^2} \left[ -v_s + \alpha \left( 1 - \frac{b^2}{r^2} \alpha \right)^{1/2} \right]$$

(2.4)

$$\vec{p} \cdot \vec{u} = -Eu^0 + p_r u^r = \frac{u_0 E}{\alpha^2} \left[ \alpha - v_s \left( 1 - \frac{b^2}{r^2} \alpha \right)^{1/2} \right]$$

(2.5)

To find the relationship between $u^0$ and $n^r$, use

$$\vec{n} \cdot \vec{n} = 1 = (n^r)^2 \left( 1 - \frac{v_s}{\alpha^3} \right) \Rightarrow (n^r)^2 = \frac{\alpha^3}{\alpha^2 - v_s^2}$$

$$\vec{u} \cdot \vec{u} = -1 = -(u^0)^2 \left( \alpha - \frac{v_s^2}{\alpha} \right) \Rightarrow (u^0)^2 = \frac{\alpha}{\alpha^2 - v_s^2}$$

and combining these two yields

$$\frac{n^r}{u^0 \alpha} = 1 \quad (2.6)$$

Finally, all the parts necessary to compute $\cos \theta$ are in the previous 3 numbered equations above (2.4-2.6).

$$\cos \theta = -\frac{\vec{n} \cdot \vec{p}}{\vec{p} \cdot \vec{u}} = \frac{\alpha \left( 1 - \frac{v_s^2}{\alpha^2} \right)^{1/2} - v_s}{\alpha - v_s \left( 1 - \frac{v_s^2}{\alpha^2} \right)^{1/2}}$$

(2.7)

Part (c)

For a photon in a circular orbit, $p_r = 0 \Rightarrow \cos \theta = -\frac{v_s}{\alpha}$. So for $r = 3M$, $\cos \theta = -3v_s$. For infalling surface, $v_s < 0$ so $\cos \theta > 0$. The photon must be emitted outwards. Notice for $|v_s| > 1/3$, $|\cos \theta| > 1$. However, a coordinate stationary observer measures the speed of the surface to be

$$\hat{v}_s = \frac{u_r}{u_0} = \frac{u^r}{\alpha u^0} = \frac{v_s}{\alpha}$$

(2.8)

For $r = 3M$, $\hat{v}_s = 3v_s$. Therefore, $|v_s|$ cannot be $> 1/3$ or else the observer sees the surface move faster than light.
Problem 3

Part (a)

Recall that for radial infall with an exterior Schwarzschild metric, that

\[ R(\eta) = \frac{R_0}{2} (1 + \cos \eta) \]
\[ \tau(\eta) = \left( \frac{R_0^3}{8M} \right)^{1/2} (\eta + \sin \eta) \]  

(3.1)

where the collapse begins at \( \eta = 0 \) with \( \tau = 0 \) and \( R = R_0 \) and ends with \( \eta = \pi, R = 0, \) and \( \pi \frac{R_0^3}{8M} = \tau_{\text{max}}. \) For homogeneous density inside \( R, \) the "mass-energy interior" to a circumferential radius \( r \) is

\[ m(r) = \int_0^r \rho 4\pi r^2 dr = \frac{4}{3} \pi r^3 \rho \]  

(3.2)

giving the relation

\[ F_i = \left( \frac{r_i(\tau)}{R(\tau)} \right)^3 \]  

(3.3)

where \( F_i \) is the fraction of mass contained within a radius \( r_i \) so then

\[ r_i(\eta) = \frac{F_i^{1/3}}{2} R_0 (1 + \cos \eta) \]  

(3.4)

See Figure 1 for a spacetime diagram illustrating this and the other parts of the problem.

Part (b)

Inside the matter, recall from class that

\[ a(\eta) = \frac{1}{2} a_{\text{max}} (1 + \cos \eta), \]
\[ \tau(\eta) = \frac{1}{2} a_{\text{max}} (\eta + \sin \eta). \]  

(3.5)

A radially outgoing photon must obey \( ds^2 = 0 \Rightarrow d\tau = a(\tau) d\chi. \) But from the equations above, \( d\tau = \frac{1}{2} a_{\text{max}} (1 + \cos \eta) d\eta = a d\eta \) so in terms of \( \eta \) the photon's equation of motion is simply

\[ \frac{d\chi}{d\eta} = 1. \]  

(3.6)
If a photon is emitted at $\eta = \eta_e, \chi = \chi_e$, then its trajectory is $\chi = \chi_e + (\eta - \eta_e)$. The circumferential radius, which is equal to the areal radius, is $r = a \sin \chi$, or

$$r(\eta) = \frac{1}{2} a_{\max}(1 + \cos \eta) \sin(\chi_e + \eta - \eta_e).$$

(3.7)

The area of a spherical pulse of light is $4\pi r^2$, so the portion of the region of trapped surfaces that lies inside of the matter is given by the values of $\eta_e, \chi_e$ satisfying

$$\frac{d}{d\eta} \left(4\pi r^2 \right) \bigg|_{\eta=\eta_e} \leq 0 \Rightarrow \frac{dr}{d\eta} \bigg|_{\eta=\eta_e} \leq 0$$

(3.8)

Plugging in the expression for $r(\eta)$ yields

$$\left[ -\frac{1}{2} a_{\max} \sin \eta \sin(\chi_e + \eta - \eta_e) + \frac{1}{2} a_{\max}(1 + \cos \eta) \cos(\chi_e + \eta - \eta_e) \right] \bigg|_{\eta=\eta_e} \leq 0$$

$$-\sin \eta_e \sin \chi_e + (1 + \cos \eta_e) \cos \chi_e \leq 0$$

$$\cos(\chi_e + \eta_e) + \cos \chi_e \leq 0$$

$$\cos(\chi_e + \eta_e) \leq \cos(\pi - \chi_e)$$

$$\chi_e + \eta_e \geq \pi - \chi_e$$

$$\eta_e \geq \pi - 2\chi_e.$$

(3.9)

Note the sign: inside the star, trapped surfaces exist outside and to the future of the curve $\eta_e + 2\chi_e = \pi$. To be inside the star we must have $\chi \leq \chi_0$, where $\chi_0$ is the $\chi$ coordinate of the surface of the star, which was calculated in class to satisfy $R_0 = a_{\max} \sin \chi_0$ and $M = \frac{1}{2} a_{\max} \sin^3 \chi_0$. Therefore, the earliest value of $\eta$ at which a trapped surface exists is

$$\eta_e = \pi - 2\sin^{-1} \left( \frac{2M}{R_0} \right)^{1/2} \equiv \eta_{AH}.$$  

(3.10)

Note that at $\eta = \eta_{AH}$, the surface of the star is at $r = 2M$. Therefore, inside the star, for $\eta < \eta_{AH}$ there is not a trapped surface, and for $\eta > \eta_{AH}$ the region of trapped surfaces is outside and to the future of the curve $\eta + 2\chi = \pi$.

So far we have said nothing about the region of trapped surfaces outside the star. There, we have the Schwarzschild metric. In outgoing Eddington-Finkelstein coordinates the equation of motion for outgoing radial photon is $\frac{dy}{dx} = 0$ where $u = t - r - 2M \ln |r/2M - 1|$. So for $r \leq 2M$

$$\frac{dt}{dr} - 1 + \frac{1}{1 - r/2M} = 0$$

$$\frac{dr}{dt} = \frac{2M}{r} (r/2M - 1)$$

(3.11)
Thus, \( \frac{dr}{dt} \leq 0 \) whenever \( r \leq 2M \), that is, everywhere outside the star and inside \( r \leq 2M \) is trapped.

The apparent horizon is the outermost boundary of trapped surfaces. For \( \eta < \eta_{AH} \) there are no trapped surfaces and thus no apparent horizon. For \( \eta > \eta_{AH} \) (at which point the radius of the surface of the star is \( 2M \)), trapped surfaces exist between \( r = 2M \) and the curve \( \eta + 2\chi = \pi \) inside the star. The outermost trapped surface, or the apparent horizon, is at \( r = 2M \).

**Part (c)**

The event horizon is the trajectory of an outgoing photon that barely reaches the surface of the matter when it reaches \( r = 2M \). Inside of the matter, use the fact that for outgoing photons \( \frac{d\chi}{d\eta} = 1 \) and that the surface hits \( r = 2M \) at \( \eta = \eta_{AH} \), to see that

\[
\chi_H = \chi_0 + \eta - \eta_{AH} \tag{3.12}
\]

This is only true for \( \eta \leq \eta_{AH} \). There is no event horizon for

\[
\eta < \eta_{AH} - \chi_0 \equiv \eta_H \tag{3.13}
\]

so the event horizon between \( \eta_H < \eta < \eta_{AH} \) is given by

\[
r_H = \frac{1}{2} a_{max} (1 + \cos \eta) \sin(\chi_0 + \eta - \eta_{AH}) \tag{3.14}
\]

For \( \eta > \eta_{AH} \), the Schwarzschild metric has an event horizon at \( r = 2M \).

**Part (d)**

For the case where \( R_o = 5M \), the apparent and event horizons start at \( \tau_{AH}/\tau_{max} = 0.8760, \tau_{H}/\tau_{max} = 0.6280 \).

<table>
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<th>( \eta )</th>
<th>( r_{1/4}/M )</th>
<th>( r_{1/2}/M )</th>
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<th>( r_{AH}/M )</th>
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Figure 1: Spacetime diagram of Oppenheimer-Snyder collapse.