Solutions Ph 236b – Week 3

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Problem 1

Part (a)

The momentum of the particle is

$$p^\mu = \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi_\mu} = g^{\mu\nu}\left(\pi_\nu - eA_\nu\right) \tag{1.1}$$

$$= \pi^\mu - eA^\mu \tag{1.2}$$

Part (b)

The equation of motion for a charged particle with the 4-vector p^μ in an electromagnetic field is given by the Lorentz force equation

$$p^\alpha p^\mu_{;\alpha} = eF^{\mu\alpha}p_\alpha \tag{1.3}$$

Part (c)

We can verify the answer from Part (b) using Hamilton's equations of motion. Eq. (3) gives

$$\frac{d\pi_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu} = \frac{1}{2}p_\alpha p_\beta g^{\lambda\alpha}g^{\sigma\beta}g_{\lambda\sigma,\mu} + eg^{\alpha\beta}p_{(\alpha}A_{\beta),\mu} \tag{1.4}$$

But $g_{\lambda\sigma} = 2\Gamma_{(\lambda\sigma)\mu}$ and $A_{\beta,\mu} = A_{\mu,\beta} + F_{\mu\beta}$, so we have

$$\frac{d\pi_\mu}{d\lambda} = p_\alpha p_\beta g^{\lambda\alpha}g^{\sigma\beta}\Gamma_{(\lambda\sigma)\mu} + eg^{\alpha\beta}\left(F_{\mu(\beta}P_{\alpha)} + A_{\mu,(\beta}P_{\alpha)}\right) \tag{1.5}$$

$$= \frac{dp_\mu}{d\lambda} + e\frac{dA_\mu}{d\lambda} \tag{1.6}$$

We note that $A_{\mu,(\beta}P_{\alpha)} = \frac{dA_\mu}{d\lambda}$ and

$$p^\alpha p_{\mu;\alpha} = p^\alpha p_{\mu,\alpha} - \Gamma^\lambda_{\mu\alpha}p_\lambda \tag{1.7}$$

So finally Eq (1.5) becomes

$$p^\alpha p^\mu_{;\alpha} = eF^{\mu\alpha}p_\alpha \tag{1.8}$$

which is the result of part (1b)

Part (d)

We know that since $\frac{\partial H}{\partial \phi} = \frac{\partial H}{\partial t} = 0$, this implies

$$\frac{d\pi_\phi}{d\lambda} = \frac{d\pi_t}{d\lambda} = 0 \tag{1.9}$$

and so π_ϕ and π_t are conserved. But

$$\pi_t = p_t + eA_t, \quad A_t = -\frac{Q}{r} \tag{1.10}$$

$$= p_t - \frac{eQ}{r} = -E \tag{1.11}$$

and

$$\pi_\phi = p_\phi + eA_\phi, \quad A_\phi = 0 \tag{1.12}$$

$$= p_\phi = L \tag{1.13}$$

so E and L are conserved.

Part (e)

We can derive Eq (6) in the problem set from the fact that the particle rest mass is conserved:

$$-\mu^2 = g_{\alpha\beta} p^\alpha p^\beta \tag{1.14}$$

$$= \left(\frac{dr}{d\lambda}\right)^2 g_{rr} + (p_0)^2 g^{00} + \frac{(p_\phi)^2}{r^2} \tag{1.15}$$

$$= \left(\frac{dr}{d\lambda}\right)^2 \frac{1}{f(r)} - \left(\frac{eQ}{r} - E\right)^2 \frac{1}{f(r)} + \frac{L^2}{r^2} \tag{1.16}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{1.17}$$

and $g_{rr} = -g^{00} = \frac{1}{f(r)}$. Rearranging will give you Eq (6).

Problem 2

Part (a)

Go to the local Lorentz frame comoving with the surface of the star so that $\vec{u} = (1, \underset{\sim}{0}), \vec{p} = (E, \underset{\sim}{p}), \vec{n} = (0, \underset{\sim}{n})$. Now $|\underset{\sim}{n}| = 1$ and $|\underset{\sim}{p}| = E = -\vec{p} \cdot u$ Then

$$\cos\theta = \frac{\underset{\sim}{n} \cdot \underset{\sim}{p}}{|\underset{\sim}{n}||\underset{\sim}{p}|} = \frac{\vec{n} \cdot \vec{p}}{|\underset{\sim}{n}||\underset{\sim}{p}|} = -\frac{\vec{n} \cdot \vec{p}}{\vec{p} \cdot u} \tag{2.1}$$

Since this final quantity is a scalar, it can be computed in any frame.

Part (b)

Without loss of generality, assume $\theta = \pi/2, p_\theta = 0$. For a photon,

$$\vec{p}\cdot\vec{p} = 0 = \frac{-1}{\alpha}(p_0)^2 + \alpha(p_r)^2 + \frac{1}{r}(p_\phi)^2 \tag{2.2}$$

where $\alpha = 1 - 2M/r$. Since $p_0 = -E$ and $p_\phi = L$ and labeling $b = L/E$, then

$$p_r = \frac{E}{\alpha}\left(1 - \frac{b^2}{r^2}\alpha\right)^{1/2} \tag{2.3}$$

Now let $v_s = \frac{dr}{dt} = u^r/u^0$. Then $\vec{u}\cdot\vec{n} = 0 = -u^0 n^0\alpha + n^r u^r/\alpha \Rightarrow n^0 = n^r v_s/\alpha^2$. Now compute the other relevant quantities

$$\vec{p}\cdot\vec{n} = -En^0 + p_r n^r = \frac{n^r E}{\alpha^2}\left[-v_s + \alpha\left(1 - \frac{b^2}{r^2}\alpha\right)^{1/2}\right] \tag{2.4}$$

$$\vec{p}\cdot\vec{u} = -Eu^0 + p_r u^r = \frac{u^0 E}{\alpha^2}\left[\alpha - v_s\left(1 - \frac{b^2}{r^2}\alpha\right)^{1/2}\right] \tag{2.5}$$

To find the relationship between u^0 and n^r, use

$$\vec{n}\cdot\vec{n} = 1 = (n^r)^2\left(\frac{1}{\alpha} - \frac{v_s}{\alpha^3}\right) \Rightarrow (n^r)^2 = \frac{\alpha^3}{\alpha^2 - v_s^2}$$

$$\vec{u}\cdot\vec{u} = -1 = -(u^0)^2\left(\alpha - \frac{v_s^2}{\alpha}\right) \Rightarrow (u^0)^2 = \frac{\alpha}{\alpha^2 - v_s^2}$$

and combining these two yields

$$\frac{n^r}{u^0\alpha} = 1 \tag{2.6}$$

Finally, all the parts necessary to compute $\cos\theta$ are in the previous 3 numbered equations above (2.4-2.6).

$$\cos\theta = -\frac{\vec{n}\cdot\vec{p}}{\vec{p}\cdot u} = \frac{\alpha\left(1 - \frac{b^2}{r^2}\alpha\right)^{1/2} - v_s}{\alpha - v_s\left(1 - \frac{b^2}{r^2}\alpha\right)^{1/2}} \tag{2.7}$$

Part (c)

For a photon in a circular orbit, $p_r = 0 \Rightarrow \cos\theta = -\frac{v_s}{\alpha}$. So for $r = 3M$, $\cos\theta = -3v_s$. For infalling surface, $v_s < 0$ so $\cos\theta > 0$. The photon must be emitted outwards. Notice for $|v_s| > 1/3, |\cos\theta| > 1$. However, a coordinate stationary observer measures the speed of the surface to be

$$\hat{v}_s = \frac{u^{\hat{r}}}{u^{\hat{0}}} = \frac{u^r}{\alpha u^0} = \frac{v_s}{\alpha} \tag{2.8}$$

For $r = 3M, \hat{v}_s = 3v_s$. Therefore, $|v_s|$ cannot be $> 1/3$ or else the observer sees the surface move faster than light.

Problem 3

Part (a)

Recall that for radial infall with an exterior Schwarzschild metric, that

$$\begin{aligned} R(\eta) =& \frac{R_0}{2}(1+\cos\eta) \\ \tau(\eta) =& \left(\frac{R_0^3}{8M}\right)^{1/2}(\eta+\sin\eta) \end{aligned} \tag{3.1}$$

where the collapse begins at $\eta = 0$ with $\tau = 0$ and $R = R_0$ and ends with $\eta = \pi, R = 0$, and $\pi\frac{R_0^3}{8M} = \tau_{max}$. For homogeneous density inside R, the "mass-energy interior" to a circumferential radius r is

$$m(r) = \int_0^r \rho 4\pi r^2 dr = \frac{4}{3}\pi r^3 \rho \tag{3.2}$$

giving the relation

$$F_i = \left(\frac{r_i(\tau)}{R(\tau)}\right)^3 \tag{3.3}$$

where F_i is the fraction of mass contained witin a radius r_i so then

$$r_i(\eta) = \frac{F_i^{1/3}}{2}R_0(1+\cos\eta) \tag{3.4}$$

See Figure 1 for a spacetime diagram illustrating this and the other parts of the problem.

Part (b)

Inside the matter, recall from class that

$$\begin{aligned} a(\eta) =& \frac{1}{2}a_{max}(1+\cos\eta), \\ \tau(\eta) =& \frac{1}{2}a_{max}(\eta+\sin\eta). \end{aligned} \tag{3.5}$$

A radially outgoing photon must obey $ds^2 = 0 \Rightarrow d\tau = a(\tau)d\chi$. But from the equations above, $d\tau = \frac{1}{2}a_{max}(1+\cos\eta)d\eta = a d\eta$ so in terms of η the photon's equation of motion is simply

$$\frac{d\chi}{d\eta} = 1. \tag{3.6}$$

If a photon is emitted at $\eta = \eta_e, \chi = \chi_e$, then its trajectory is $\chi = \chi_e + (\eta - \eta_e)$. The circumferential radius, which is equal to the areal radius, is $r = a \sin\chi$, or

$$r(\eta) = \frac{1}{2} a_{max}(1 + \cos\eta)\sin(\chi_e + \eta - \eta_e). \tag{3.7}$$

The area of a spherical pulse of light is $4\pi r^2$, so the portion of the region of trapped surfaces that lies inside of the matter is given by the values of η_e, χ_e satisfying

$$\frac{d}{d\eta}(4\pi r^2)\bigg|_{\eta=\eta_e} \leq 0 \Rightarrow \frac{dr}{d\eta}\bigg|_{\eta=\eta_e} \leq 0 \tag{3.8}$$

Plugging in the expression for $r(\eta)$ yields

$$\begin{aligned}\left[-\frac{1}{2}a_{max}\sin\eta\sin(\chi_e + \eta - \eta_e) + \frac{1}{2}a_{max}(1 + \cos\eta)\cos(\chi_e + \eta - \eta_e)\right]\bigg|_{\eta=\eta_e} &\leq 0 \\ -\sin\eta_e\sin\chi_e + (1 + \cos\eta_e)\cos\chi_e &\leq 0 \\ \cos(\chi_e + \eta_e) + \cos\chi_e &\leq 0 \\ \cos(\chi_e + \eta_e) &\leq \cos(\pi - \chi_e) \\ \chi_e + \eta_e &\geq \pi - \chi_e \\ \eta_e &\geq \pi - 2\chi_e.\end{aligned} \tag{3.9}$$

Note the sign: inside the star, trapped surfaces exist *outside* and *to the future of* the curve $\eta_e + 2\chi_e = \pi$. To be inside the star we must have $\chi \leq \chi_o$, where χ_0 is the χ coordinate of the surface of the star, which was calculated in class to satisfy $R_o = a_{max}\sin\chi_o$ and $M = \frac{1}{2}a_{max}\sin^3\chi_o$. Therefore, the earliest value of η at which a trapped surface exists is

$$\eta_e = \pi - 2\sin^{-1}\left(\frac{2M}{R_o}\right)^{1/2} \equiv \eta_{AH}. \tag{3.10}$$

Note that at $\eta = \eta_{AH}$, the surface of the star is at $r = 2M$. Therefore, inside the star, for $\eta < \eta_{AH}$ there is not a trapped surface, and for $\eta > \eta_{AH}$ the region of trapped surfaces is *outside* and *to the future of* the curve $\eta + 2\chi = \pi$.

So far we have said nothing about the region of trapped surfaces *outside* the star. There, we have the Schwarzchild metric. In outgoing Eddington-Finkelstein coordinates the equation of motion for outgoing radial photon is $\frac{d\underset{\sim}{u}}{dr} = 0$ where $\underset{\sim}{u} = t - r - 2M\ln|r/2M - 1|$. So for $r \leq 2M$

$$\begin{aligned}\frac{dt}{dr} - 1 + \frac{1}{1 - r/2M} &= 0 \\ \frac{dr}{dt} &= \frac{2M}{r}(r/2M - 1)\end{aligned} \tag{3.11}$$

Thus, $\frac{dr}{dt} \leq 0$ whenever $r \leq 2M$, that is, everywhere *outside* the star and inside $r \leq 2M$ is trapped.

The apparent horizon is the outermost boundary of trapped surfaces. For $\eta < \eta_{AH}$ there are no trapped surfaces and thus no apparent horizon. For $\eta > \eta_{AH}$ (at which point the radius of the surface of the star is $2M$), trapped surfaces exist between $r = 2M$ and the curve $\eta + 2\chi = \pi$ inside the star. The outermost trapped surface, or the apparent horizon, is at $r = 2M$.

Part (c)

The event horizon is the trajectory of an outgoing photon that barely reaches the surface of the matter when it reaches $r = 2M$. Inside of the matter, use the fact that for outgoing photons $\frac{d\chi}{d\eta} = 1$ and that the surface hits $r = 2M$ at $\eta = \eta_{AH}$, to see that

$$\chi_H = \chi_o + \eta - \eta_{AH} \tag{3.12}$$

This is only true for $\eta \leq \eta_{AH}$. There is no event horizon for

$$\eta < \eta_{AH} - \chi_0 \equiv \eta_H \tag{3.13}$$

so the event horizon between $\eta_H < \eta < \eta_{AH}$ is given by

$$r_H = \frac{1}{2} a_{max}(1 + \cos\eta)\sin(\chi_o + \eta - \eta_{AH}) \tag{3.14}$$

For $\eta > \eta_{AH}$, the Schwarzschild metric has an event horizon at $r = 2M$.

Part (d)

For the case where $R_o = 5M$, the apparent and event horizons start at $\tau_{AH}/\tau_{max} = 0.8760$, $\tau_H/\tau_{max} = 0.6280$.

τ/τ_{max}	η	$r_{1/4}/M$	$r_{1/2}/M$	R/M	r_{AH}/M	r_H/M
0.0	0	3.1498	3.9686	5.0	—	—
0.2	.3168	3.0714	3.8699	4.8756	—	—
0.4	.6508	2.8279	3.5630	4.4890	—	—
0.6	1.029	2.3877	3.0084	3.7903	—	—
0.8	1.515	1.6630	2.0954	2.6399	—	0.8421
1.0	3.142	0	0	0	2.0	2.0

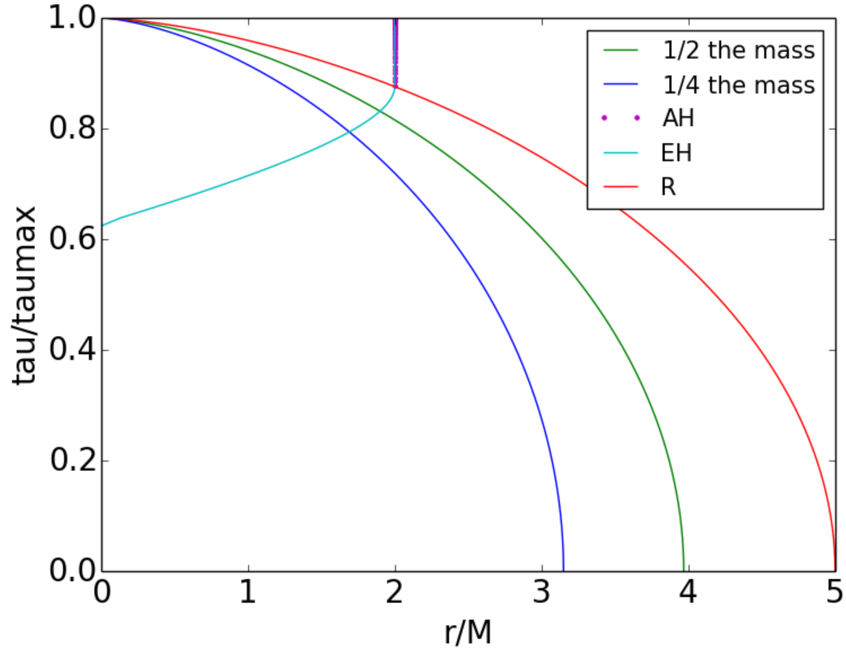

Figure 1: Spacetime diagram of Oppenheimer-Snyder collapse.