Solutions Ph 236b – Week 1

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Problem 1

Part (a)

The stress-energy tensor for a perfect fluid is given by

$$T^{\alpha\beta} = (P+\rho)u^\alpha u^\beta + P g^{\alpha\beta}, \tag{1.1}$$

and perturbations are of the form $P = P_0 + \delta P, \rho = \rho_0 + \delta\rho, u^i = (1, v_i)$. Now look at the different components of $T^{\alpha\beta}{}_{;\beta} = 0$ to get an equation of the perturbation variables (to 1st order). For the case $\alpha = 0$:

$$\begin{aligned}(P+\rho)u^0 u^\beta{}_{;\beta} + (P+\rho)_{;\beta}u^0 u^\beta + (P+\rho)u^0{}_{;\beta}u^\beta + P_{;\beta}g^{\alpha\beta} + Pg^{0\beta}{}_{;\beta} &= 0\\ (\delta P + \delta\rho)_{;0} + (P_0+\rho_0)v^i{}_{;i} - \delta P_{;0} &= 0\\ v^i{}_{;i} = \nabla\cdot \underset{\sim}{v} = -\frac{\partial(\delta\rho)}{\partial t}(P_0+\rho_0)^{-1}.\end{aligned} \tag{1.2}$$

For $\alpha = j$ (where $j = 1, 2, 3$):

$$\begin{aligned}(P+\rho)u^j u^\beta{}_{;\beta} + (P+\rho)u^j{}_{;\beta}u^\beta + (P+\rho)_{;\beta}u^j u^\beta + P_{;\beta}g^{\alpha\beta} + Pg^{j\beta}{}_{;\beta} &= 0\\ (P+\rho)u^j{}_{;0} + P_{;j} &= 0\\ \frac{\partial \underset{\sim}{v}}{\partial t} = -\frac{\nabla(\delta P)}{(P_0+\rho_0)^{-1}}.\end{aligned} \tag{1.3}$$

Part (b)

Taking the equations of motion from above and combining them gives

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla\cdot\underset{\sim}{v}) &= \nabla\cdot\frac{\partial\underset{\sim}{v}}{\partial t}\\ \frac{\partial}{\partial t}\left(-\frac{\partial(\delta\rho)}{\partial t}(P_0+\rho_0)^{-1}\right) &= \nabla\cdot\left(-\frac{\nabla(\delta P)}{(P_0+\rho_0)^{-1}}\right)\\ \frac{\partial^2(\delta\rho)}{\partial t^2} &= \nabla^2(\delta P).\end{aligned} \tag{1.4}$$

Let $v_s^2 = \frac{\partial P}{\partial\rho}\big|_s$. Substituting $\delta P = \frac{\partial P}{\partial \rho}\big|_s \delta\rho = v_s^2\delta\rho$ into the differential equation yields

$$\nabla^2(\delta\rho) - \frac{1}{v_s^2}\frac{\partial^2(\delta\rho)}{\partial t^2} = 0. \tag{1.5}$$

This is just the wave equation where the constant coefficient is related to the sound speed. The perturbations travel at a speed v_s.

Problem 2

Part (a)

Comparing the two froms for the metric, there are two equations which relate the Schwarzchild and isotropic coordinates:

$$\begin{aligned} e^{2\Lambda} dr^2 =& e^{2\mu} d\bar{r}^2, \\ r^2 =& e^{2\mu} \bar{r}^2. \end{aligned} \tag{2.1}$$

Dividing these two equations by each other and taking the square root leaves

$$e^{\Lambda} \frac{dr}{r} = \frac{d\bar{r}}{\bar{r}}. \tag{2.2}$$

Integrate this equation to get $\bar{r}$ in terms of r and Λ,

$$\bar{r} = C \exp\left(\int \frac{e^{\Lambda}}{r} dr \right), \tag{2.3}$$

Where C is a constant of integration. Since $\bar{r}$ is in terms of r and Λ, μ can now be expressed as $e^{2\mu} = r^2/\bar{r}^2$.

Part (b)

In vacuum, $e^{\Lambda} = (1 - 2M/r)^{-1/2}$. Toss this into the integral for $\bar{r}$ so that

$$\begin{aligned} \bar{r} &= C \exp\left(\int \frac{dr}{(r^2 - 2Mr)^{1/2}} \right) \\ &= C(\sqrt{r} + \sqrt{r - 2M})^2, \end{aligned} \tag{2.4}$$

assuming $r \geq 2M$. This expression can be inverted to get

$$r = \frac{(\bar{r}/C + 2M)^2}{4\bar{r}/C} = \frac{\bar{r}}{4C} (1 + 2MC/\bar{r})^2. \tag{2.5}$$

Impose the condition that $r = \bar{r}$ as they both grow to infinity to set the constant of integration $C = 1/4$ leaving

$$\begin{aligned} r &= \bar{r}(1 + M/2\bar{r})^2, \\ e^{2\mu} &= (1 + M/2\bar{r})^4. \end{aligned} \tag{2.6}$$

Part (c)

Looking at the metric for isotropic coordinates, for a constant t and $\bar{r}$, the area element is given by $e^{2\mu}\bar{r}^2 d\Omega^2$. Integrate over the angular coordinates to get the area of the sphere,

$$\begin{aligned} A =& e^{2\mu}\bar{r}^2 \int d\Omega^2 \\ =& (1 + M/2\bar{r})^4 \bar{r}^2 \int d\theta \sin\theta d\phi \\ =& 4\pi\bar{r}^2 (1 + M/2\bar{r})^4 \end{aligned} \tag{2.7}$$

Part (d)

In constructing the embedding diagram, note that from Part (b) that the mapping from r to $\bar{r}$ is double-valued and that the coordinate $\bar{r}$ only describes the region for $r \geq 2M$. So an embedding diagram for a surface of constant t requires that

$$\begin{aligned} ds^2 =& dz^2 + dr^2 + r^2 d\phi^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2\phi^2 \\ =& (1 - 2M/r)^{-1} dr^2 + r^2 d\phi^2 \end{aligned} \tag{2.8}$$

which has a solution of

$$z = [8M(r - 2M)]^{1/2} \tag{2.9}$$

So the diagram in the $r - z$ plane is a sideways parabola opening to the right, with $\bar{r} = M/2$ at $z = 0$. As r goes to infinity, the top half of the parabola is where $\bar{r}$ goes to infinity and the bottom half is where $\bar{r}$ goes to 0. Thus $\bar{r} > M/2$ represents our universe, i.e. region I of the Kruskal diagram of a Schwarzschild black hole, and $\bar{r} < M/2$ represents the "other universe", i.e. region III of the Kruskal diagram.

Problem 3

Part (a)

Substitute the expression for the phase space distribution function into each of the expressions so

$$n = \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp \tag{3.1}$$

$$\rho = \frac{8\pi}{h^3}\int_0^{p_F} (p^2+m^2)^{1/2} p^2 dp \tag{3.2}$$

$$P = \frac{8\pi}{3h^3}\int_0^{p_F} v p^3 dp \tag{3.3}$$

Part (b)

Given that the rest mass of the fermions is much less than the Fermi momentum p_F, then neglecting the rest mass gives the approximations that $(p^2+m^2)^{1/2} \approx p$ and $v \approx 1$. Substituting that above into the integrals for P and ρ,

$$\begin{aligned} \rho =& \frac{8\pi}{h^3}\int_0^{p_F} p^3 dp = \frac{8\pi}{4h^3} p_F^4 \\ P =& \frac{8\pi}{3h^3}\int_0^{p_F} p^3 dp = \frac{8\pi}{12h^3} p_F^4 \end{aligned} \tag{3.4}$$

From here, notice that then $P(\rho) = \frac{\rho}{3}$.

Part (c)

From the results of Problem 1b above, the sound speed is given by $v_s^2 = \frac{\partial P}{\partial \rho}$. Since

$$\frac{\partial P}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{\rho}{3}\right) = \frac{1}{3} \tag{3.5}$$

the sound speed, $v_s = \sqrt{1/3}$.

Part (d)

Consider the two equations of stellar structure discussed in class,

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{3.6}$$

$$\frac{dP}{dr} = \frac{-(P+\rho)(4\pi r^3 P + m)}{r(r-2m)} \tag{3.7}$$

Throw the suggested solution of $m(r) = 3r/14$ into the first equation to get

$$\rho(r) = (3/14)(4\pi r^2)^{-1} \tag{3.8}$$

Now plug in the equation of state found in part b, $P = \rho/3$, into the second structure equation

$$\frac{d\rho}{dr} = -\frac{4\rho(4\pi r^3\rho/3 + m)}{r(r-2m)} \tag{3.9}$$

Mix in the equations for $m(r)$ and $\rho(r)$ to and simplify to show that they do indeed statisfy the equations of stellar structure.

Part (e)

From the previous part, $\rho(r) = (3/14)(4\pi r^2)^{-1}$ and the equation of state is $P(r) = (1/14)(4\pi r^2)^{-1}$. To get $n(r)$, consider the integrals from part (a) for this case.

$$\begin{aligned} n =& \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp = \frac{8\pi}{3h^3}p_F^3 \\ p =& \frac{8\pi}{h^3}\int_0^{p_F} p^3 dp = \frac{8\pi}{4h^3}p_F^4 \end{aligned} \tag{3.10}$$

Eliminating p_F and solving for $n(r)$ gives

$$n(r) = \frac{8\pi}{3h^3}\left(\frac{4h^3\rho}{8\pi}\right)^{3/4} \tag{3.11}$$

and finally, substituting in for ρ reduces the result to

$$n(r) = r^{-3/2}\frac{8\pi}{3h^3}\left(\frac{4h^3}{8\pi}\frac{3}{14}\frac{1}{4\pi}\right)^{3/4} = \frac{K}{r^{3/2}}, \tag{3.12}$$

where

$$K = \frac{8\pi}{3h^3}\left(\frac{4h^3}{8\pi}\frac{3}{14}\frac{1}{4\pi}\right)^{3/4} \tag{3.13}$$

is a constant.

Part (f)

The total number of particles inside radius r is

$$\begin{aligned} N(r) =& \int_0^r n(r)d(\text{proper volume}) \\ =& \int_0^r n(r)e^{\Lambda}4\pi r^2 dr \\ =& \frac{14\pi K}{3}r^{3/2} \end{aligned} \tag{3.14}$$

where

$$e^{2\Lambda} = g_{rr} = (1 - 2m(r)/r)^{-1} = 7/4 \tag{3.15}$$

Thus the total number of particles N is finite for all r.

Part (g)

The 3-geometry of a t =constant hypersurface has a metric

$$ds^2 = g_{rr}dr^2 + r^2 d\Omega^2 = (7/4)dr^2 + r^2 d\Omega^2 \tag{3.16}$$

so then the embedding equation for constant θ, ϕ is

$$\begin{aligned} ds^2 = (7/4)dr^2 &= dr^2 + dz^2 \\ \Rightarrow z &= \pm(3/4)^{1/2} r \end{aligned} \tag{3.17}$$

Thus, the embedding diagram will be two lines with slopes of $\pm(3/4)^{1/2}$ that intersect at $r = 0$

Problem 4

The line element of Schwarzchild metric is

$$ds^2 = -e^{2\Phi}dt^2 + (1 - 2m(r)/r)^{-1}dr^2 + r^2 d\Omega^2 \tag{4.1}$$

From this line element, the measured area of a sphere of coordinate radius r is $A(r) = 4\pi r^2$. The gradient of a scalar is a 1-form

$$\nabla A(r) \equiv \widetilde{dA} = 8\pi r \widetilde{dr} \tag{4.2}$$

From this, create an invarient quantity, namely

$$\begin{aligned} \widetilde{dA} \cdot \widetilde{dA} &= 16\pi^2 r^2 \widetilde{dr} \cdot \widetilde{dr} = 16\pi^2 r^2 (1 - 2m/r) \\ \Rightarrow m(r) &= \frac{(A/\pi)^{1/2}}{4} \left(1 - \frac{\widetilde{dA} \cdot \widetilde{dA}}{16\pi A}\right) \end{aligned} \tag{4.3}$$