Physics 236b assignment, Week 9:

(March 3, 2016. Due on March 10, 2016)

1. **Scalar waves in curved spacetime** [25 points]

   The wave equation for a scalar field ϕ reads $g^{ab}\nabla_a\nabla_b\phi = 0$, where ∇_a is a 4-d covariant derivative.

   Assume a metric in 3+1 form

   $$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{1}$$

   (a) Define

   $$\Pi \equiv -\frac{1}{\alpha}\mathcal{L}_{\alpha n}\phi, \tag{2}$$

   $$\Phi_a \equiv D_a\phi, \tag{3}$$

   so in components

   $$\Pi = -\frac{1}{\alpha}\left(\partial_0 - \beta^i\partial_i\right)\phi, \tag{4}$$

   $$\Phi_k = \partial_k\phi, \tag{5}$$

   and show that the scalar wave equation can be written in the form

   $$(\partial_0 - \mathcal{L}_\beta)\Pi + \alpha\gamma^{ij}\partial_i\Phi_j = \alpha\left(K\Pi - \Phi^k\partial_k\ln\alpha + \Phi_k\Gamma^{kj}{}_j\right), \tag{6}$$

   where ∂_a is an ordinary partial derivative, K is the trace of the extrinsic curvature, and $\Gamma^k{}_{ij}$ is the 3-dimensional connection compatable with the 3-metric γ_{ij}.

   Hint: while this can be done by computing the 4-d Christoffel symbols from Eq. (1), the calculation has far fewer terms if you instead compute $\mathcal{L}_{\alpha n}\Pi$ (which equals $(\partial_0 - \mathcal{L}_\beta)\Pi$) in terms of objects like D_a, n_a, $\gamma_a{}^b$, K_{ab}, etc, and only specialize to a basis at the end.

(b) Show that

$$(\partial_0 - \mathcal{L}_\beta)\Phi_i + \alpha\partial_i\Pi = -\Pi\partial_i\alpha. \tag{7}$$

(c) Equations (6), (7), and (4) make up a system of 1st order partial differential equations for the 5 variables Π, Φ_i, and ϕ, assuming that the metric quantities (γ_{ij}, K_{ij}, α, β^i, and their derivatives) are known. This system can be written in the form

$$\partial_0 u^a + A^{ia}{}_b \partial_i u^b = B^a, \tag{8}$$

where u^a is a column vector containing $\phi, \Pi, \Phi_x, \Phi_y, \Phi_z$, and the indices a and b index this column vector and are not tensor indices. The matrices $A^{ia}{}_b$ and the column vector B^a can depend on u^a but not on derivatives of u^a.

Given a unit vector ξ_i, find the characteristic fields and characteristic speeds, in terms of ξ_i, Π, Φ_i, ϕ, β^i, α, and γ_{ij}. There should be five characteristic fields and five corresponding characteristic speeds. (In class I used n_i instead of ξ_i for the spatial unit vector that defines characteristic fields and characteristic speeds. Here I'm using ξ_i to avoid confusion with the timelike normal to the slice n^a.)

(d) Show that the system of equations in part 1c is strongly hyperbolic.

(e) Show that the system of equations in part 1c is symmetric hyperbolic, by explicity constructing a positive definite symmetrizer S_{ab} such that $S_{ac}A^{ic}{}_b$ is symmetric in a and b.

(f) Suppose that $\gamma_{ij} = \delta_{ij}$, $K_{ij} = 0$, $\alpha = 1$, and $\beta^i = (2, 0, 0)$. This represents a flat metric in a moving coordinate system. Suppose that you were to solve the system of equations from part 1c inside a box aligned with the x, y, z coordinates. What quantities require boundary conditions on the positive x face of the box? The negative x face? The positive z face?

2. **Constraint damping for scalar waves** [15 points]

   (a) Equation (5) can be thought of as a constraint that must be satisfied at the initial time, and during the evolution. Let

$$C_i \equiv \partial_i \phi - \Phi_i, \tag{9}$$
$$C_{ij} \equiv \partial_{[i} \Phi_{j]}. \tag{10}$$

   Then $C_i = 0$ is equivalent to Equation (5), and $C_{ij} = 0$ must be true if $C_i = 0$. Treating C_i and C_{ij} as independent variables, use Equations (6), (7), and (4) to write down evolution equations for C_i and C_{ij}. Show from these evolution equations that if $C_i = 0$ and $C_{ij} = 0$ initially, then $C_i = 0$ and $C_{ij} = 0$ for all time. However, if C_i or C_{ij} are nonzero (say because of numerical error), they will in general remain nonzero.

   (b) Modify Eq. (7) as follows:

$$(\partial_0 - \mathcal{L}_\beta)\Phi_i + \partial_i \Pi = -\Pi \partial_k \alpha + \eta \alpha C_i. \tag{11}$$

   Here we have added a constraint to the right-hand side of the equation, with a constant coefficient η. We are free to do this because any solution of Eq. (11) that satisfies $C_i = 0$ also satisfies Eq. (7). Using Eq. (11), construct the evolution equations for C_i and C_{ij}. Show that with the new evolution equations, if C_i or C_{ij} are nonzero, they will be damped to zero with a timescale $\alpha\eta$, if $\eta > 0$.

3. **Generalized Harmonic formulation** [10 points]

   Here we will derive some generalized harmonic results stated in class but not proven.

   (a) Show that $\nabla_\alpha \nabla^\alpha x^\beta = -\Gamma^\beta{}_{\alpha\gamma} g^{\alpha\gamma}$, where ∇_α and $\Gamma^\alpha{}_{\beta\gamma}$ are the 4-d covariant derivative and connection compatable with the 4-metric $g_{\alpha\beta}$. Note that x^β is to be regarded as a set of 4 scalar functions (coordinates) labeled by the index β; x^β is not a vector.

(b) Given the ADM metric, Eq. (1), and assuming generalized harmonic coordinates $\nabla_\alpha \nabla^\alpha x^\beta = H^\beta$, show that

$$\partial_0 \alpha - \beta^k \partial_k \alpha = -\alpha(H_0 - \beta^i H_i + \alpha K), \tag{12}$$

$$\partial_0 \beta^i - \beta^k \partial_k \beta^i = \alpha^2 \gamma^{ij}(H_j + \gamma^{kl}\Gamma_{jkl} - \partial_j \ln \alpha), \tag{13}$$

where Γ_{jkl} is the *three-dimensional* connection compatable with the 3-metric γ_{ij}, and K is the trace of the extrinsic curvature.

(c) If n^α is the timelike normal, so that the components of n^α are $(1, \beta^i)/\alpha$, show that

$$n^\alpha H_\alpha = n^\alpha \partial_\alpha \ln\left(\frac{\gamma^{1/2}}{\alpha}\right) - \frac{1}{\alpha}\partial_k \beta^k. \tag{14}$$

This equation motivates the choice of time coordinate in the damped harmonic gauge.

4. **BSSN relations** [10 points]

(a) The BSSN system defines $\bar{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$ where γ_{ij} is the 3-metric and ϕ is the conformal factor. Show that if we demand $\bar{\gamma} = 1$ (where $\bar{\gamma}$ is the determinant of $\bar{\gamma}_{ij}$), then

$$\phi = \frac{1}{12}\ln \gamma, \tag{15}$$

where γ is the determinant of γ_{ij}.

(b) Using the ADM evolution equations, derive the BSSN evolution equations for the conformal factor and the conformal metric

$$\partial_t \phi - \beta^i \partial_i \phi = \frac{1}{6}\partial_i \beta^i - \frac{1}{6}\alpha K, \tag{16}$$

$$\partial_t \bar{\gamma}_{ij} - \beta^k \partial_k \bar{\gamma}_{ij} = -2\alpha \bar{A}_{ij} + 2\bar{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k \beta^k, \tag{17}$$

Where $\bar{A}_{ij}$ is defined such that

$$K_{ij} = e^{4\phi} \left(\bar{A}_{ij} + \frac{1}{3} \bar{\gamma}_{ij} K \right). \tag{18}$$