Physics 236b assignment, Week 8:

(Feb 25, 2016. Due on March 4, 2016)

1. **Warp drive** [20 points]

   Consider the metric

$$ds^2 = -dt^2 + (dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2, \tag{1}$$

where

$$\begin{aligned} x_s(t) &= \text{an arbitrary function of coordinate time}, &(2)\\ v_s(t) &= \frac{dx_s}{dt}, &(3)\\ r_s(t) &= \left((x - x_s(t))^2 + y^2 + z^2\right)^{1/2}, &(4)\\ f(r_s) &= \frac{\tanh\left(\sigma(r_s + R)\right) - \tanh\left(\sigma(r_s - R)\right)}{2\tanh(\sigma R)}, &(5)\\ & &(6) \end{aligned}$$

and where R and σ are constants. The function $f_s(r)$ describes a spherical "warp bubble" of coordinate radius R that moves with coordinate velocity $v_s(t)$. Notice that $v_s(t)$ is the derivative of an arbitrary function, so it could be larger than the speed of light. The constant σ determines the width of the warp bubble. For $\sigma \to \infty$, $f(r_s)$ is 1 inside $r_s = R$ and 0 outside $r_s = R$; for large but finite σ there is a region around $r_s = R$ where $f(r_s)$ changes rapidly from 1 to zero.

   (a) Compute the 3-dimensional metric, the lapse function, the shift vector, and the extrinsic curvature for this spacetime, for slices of constant t.

   (b) What is the normal vector $\vec{n}$ to the hypersurfaces of constant t?

(c) Show that all observers who have 4-velocity $\vec{u} = \vec{n}$ travel along geodesics.

(d) Show that spacetime is flat outside the warp bubble, so that distant observers see the warp bubble move at speed $v_s(t)$.

(e) Now consider a spaceship at the center of the warp bubble, i.e. at $x(t) = x_s(t)$. Show that this spaceship moves along a geodesic, and experiences no time dilation with respect to distant observers.

Notice that one could set $v_s(t)$ to zero at early times (so that the metric is flat everywhere), then gradually ramp it up to arbitrarily large values, and then return v_s to zero at late times (so that the metric is flat again). Thus, the spaceship could travel a large distance in flat space at an essentially arbitrary speed, even though the spaceship's proper acceleration is always zero.

(f) Now the catch: If this metric satisfies Einstein's equations, there must be a $T^{\mu\nu}$ that generates the required curvature. Compute the energy density $T^{\mu\nu} n_\mu n_\nu$ observed by someone whose four-velocity is equal to the hypersurface normal $\vec{n}$. Comment on the result.

2. **Lie derivative of tensor density** [15 points]

(a) Show that

$$\mathcal{L}_v \sqrt{\gamma} = \sqrt{\gamma} D_a v^a \tag{7}$$

where γ_{ab} is a spatial metric, γ is its determinant, v^a is some vector field, and D_a is the covariant derivative compatable with γ_{ab}. Note that even though $\sqrt{\gamma}$ doesn't have indices, it is not a scalar. Hint: Use $\delta \ln \det A = \mathrm{Tr}(A^{-1} \delta A)$.

(b) Show that

$$\mathcal{L}_{\alpha n} \ln \sqrt{\gamma} = -\alpha K, \tag{8}$$

where n is the normal to a 3-dimensional slice, γ_{ij} is the 3-metric, α is the lapse, and K is the trace of the extrinsic curvature.

3. **Conformal decomposition** [20 points]

   Suppose $\gamma_{ab} = \psi^4 \bar{\gamma}_{ab}$, where γ_{ab} is the 3-dimensional metric, ψ is a function called the *conformal factor*, and $\bar{\gamma}_{ab}$ is called the *conformal 3-metric.*

   (a) Show that the Ricci tensor R_{ab} of the 3-metric γ_{ab} is related to the Ricci scalar $\bar{R}_{ab}$ of the conformal 3-metric $\bar{\gamma}_{ab}$ by

   $$\begin{aligned} R_{ab} =& \bar{R}_{ab} \\ & - 2\bar{D}_a \bar{D}_b \ln\psi - 2\bar{\gamma}_{ab}\bar{\gamma}^{cd}\bar{D}_c\bar{D}_d \ln\psi \\ & + 4\bar{D}_a \ln\psi \bar{D}_b \ln\psi - 4\bar{\gamma}_{ab}\bar{\gamma}^{cd}\bar{D}_c \ln\psi \bar{D}_d \ln\psi, \end{aligned} \tag{9}$$

   where $\bar{D}_a$ is the spatial covariant derivative compatable with the conformal metric $\bar{\gamma}_{ab}$.

   (b) Show that the Ricci scalar R of the 3-metric γ_{ab} is related to the Ricci scalar $\bar{R}$ of the conformal 3-metric $\bar{\gamma}_{ab}$ by

   $$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{\gamma}^{ab}\bar{D}_a\bar{D}_b\psi. \tag{10}$$

   (c) Suppose that γ_{ab} is *conformally flat*, which means that $\bar{\gamma}_{ab} = \delta_{ab}$, the flat (spatial) metric. Write the Hamiltonian constraint as a differential equation for the conformal factor ψ.

4. **Conformally-flat static spacetime** [15 points]

   Consider a static, asymptotically-flat vacuum spacetime with a conformally flat spatial metric $\gamma_{ij} = \psi^4\delta_{ij}$. (See problem 3c). A spatial slice is called *time-symmetric* if $K_{ij} = 0$ Assume for simplicity that the spatial slices are *all* time-symmetric.

(a) Use the Hamiltonian constraint to show that

$$\psi = 1 + k/r, \tag{11}$$

where k is a constant and r is the radial coordinate.

(b) Use the evolution equation for the trace of the extrinsic curvature to find that the lapse function satisfies

$$\alpha^2 = \frac{(1 - k/r)^2}{(1 + k/r)^2} \tag{12}$$

(c) Show that this spacetime is Schwarzschild (but not in the usual coordinates). Find the usual Schwarzschild radial coordinate r_s in terms of r, and find the Schwarzschild mass M in terms of k.