Physics 236b assignment, Week 7:

(Feb 18, 2016. Due on Feb 25, 2016)

1. **Extrinsic curvature of 2D surfaces** [15 points] In class we talked about the extrinsic curvature of a 3D surface embedded in a 4D spacetime. However, the extrinsic curvature is defined the same way for a 2D surface embedded in a 3D spacetime, e.g.

$$K_{ab} = -\gamma_a^c \gamma_b^d \nabla_c n_d, \tag{1}$$

where n_d is the normal to the surface.

The eigenvalues and eigenvectors of the extrinsic curvature tensor are called the principal curvatures and the principal directions. Find the principal curvatures and principal directions for the following surfaces embedded in 3D Euclidean space:

   (a) A sphere: $x^2 + y^2 + z^2 = a^2$

   (b) A cylinder: $x^2 + y^2 = a^2$

   (c) A quadratic surface: $z = \frac{1}{2}(ax^2 + 2bxy + cy^2)$ (compute at the origin only).

2. **More on extrinsic curvature of 2D surfaces** [15 points] Show that the scalar curvature of a 2D surface in flat 3D space is

$$^{(2)}R = \frac{2}{\rho_1 \rho_2}, \tag{2}$$

where ρ_1 and ρ_2 are the principal radii of curvature of the 2D surface (defined in Problem 1). What is the analogous formula for a 3D surface embedded in flat 4D space? Note: The Gauss equation reads

$$^{(n-1)}R_{abcd} = \gamma_a^e \gamma_b^f \gamma_c^g \gamma_d^{h\,(n)} R_{efgh} + 2K_{b[d}K_{c]a}, \tag{3}$$

which has the opposite sign on the extrinsic curvature than the equation given in class, because here the normal satisfies $\vec{n} \cdot \vec{n} = +1$, whereas in class we were concerned with a normal vector satisfying $\vec{n} \cdot \vec{n} = -1$.

3. **Separation rate of curves** [10 points]

   Let C_Σ be a curve in a 3-surface Σ, and let $\vec{u}$ be the tangent vector to C_Σ. Let C be a geodesic in the 4D space in which Σ is embedded. Let $\vec{n}$ be the unit normal to Σ. Suppose that C_Σ and C are tangent at some point q in Σ. Then the vector $\xi^a = u^b \nabla_b u^a$ is the acceleration of C_Σ, i.e. the rate at which C and C_Σ separate. Show that at the point q,

   $$n^a \xi_a = K_{ab} u^a u^b, \tag{4}$$

   where K_{ab} is the extrinsic curvature of Σ.

4. **Soap bubble** [15 points]

   Neglecting gravity, the potential energy due to surface tension of a soap film is proportional to its area. Thus in equilibrium a soap film spanning a fixed closed wire loop will assume a shape of minimum area. Show that this implies that the surface has $K^i_i = 0$, where K_{ij} is the extrinsic curvature. Hint: choose coordinates such that $x^3 = 0$ on the surface, i.e. the normal is $\vec{n} = \partial/\partial x^3$. Then consider varying the area of the surface by displacing it by a distance δx^3 along the normal.