Physics 236b assignment, Week 5:

(Feb 4, 2016. Due on Feb 11, 2016)

1. **Polar orbits in Kerr spacetime** [20 points]

   (a) Show that there exist "quasicircular" polar orbits in Kerr spacetime, i.e. orbits that pass alternately over the north and south poles at a fixed radial coordinate distance.

   (b) What is the minimum possible radius of such an orbit?

   (c) What is $d\phi/dt$ for such an orbit?

   (d) Is the orbit stable under radial perturbations?

2. **Energy from black hole collisions** [10 points]

   Two Kerr black holes, with masses M_1 and M_2 and spins a_1 and a_2, collide to form a new black hole with mass M_3 and a_3. Gravitational waves are emitted in the process.

   (a) Suppose the initial black holes have equal masses and opposite spins ($M_1 = M_2$, $a_1 = -a_2$), and the final black hole is Schwarzschild ($a_3 = 0$). According to Hawking's area theorem, what is the maximum amount of energy (expressed as a fraction of the total initial mass) that can be radiated away in the form of gravitational waves?

   (b) What is the answer to part 2a if the original black holes have maximal spin?

   (c) Show that for arbitrary $M_1, M_2, a_1, a_2, M_3, a_3$, the maximum fraction of the initial mass that is radiated as gravitational waves is not larger than the answer to part 2b.

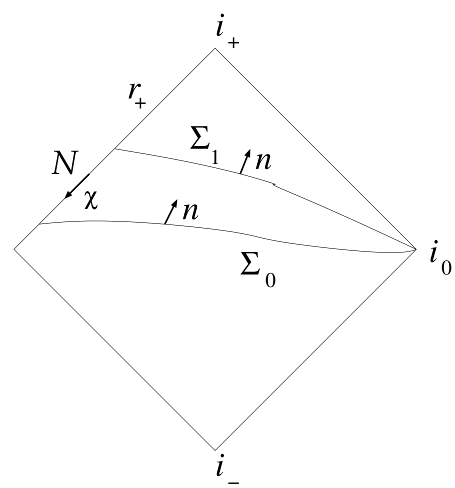
3. **Can you make a naked singularity?** [10 points] Show that by throwing particles into a Kerr black hole, it is not possible to increase the spin above $|a| = M$.

4. **Negative energy orbits** [15 points] In Kerr spacetime, can there be particles with negative energy in circular equatorial orbits?

5. **Superradiance** [20 points]

   Consider a scalar field scattering off of a Kerr black hole. One often thinks of an incident wave, a transmitted wave (which in this case goes down the black hole), and a reflected wave. Normally the reflected wave has less energy than the transmitted wave, but we will show that in some cases, the reflected wave *gains* energy from the black hole, in analogy with the Penrose process.

   (a) Suppose $\vec{\xi}$ is a Killing vector and $T^{\alpha\beta}$ is a stress-energy tensor. Show that if $J^{\alpha} = -T^{\alpha\beta}\xi_{\beta}$, then $J^{\alpha}{}_{;\alpha} = 0$.



   (b) Now suppose that $\vec{\xi}$ is a timelike Killing vector, and assume $T^{\alpha\beta} \to 0$ at spatial infinity i_0. Define E_0 and E_1 as the total (nongravitational) energy on the spacelike hypersurfaces Σ_0 and Σ_1 in the Penrose diagram, respectively. Both Σ_0 and Σ_1 have future-pointing normal vectors. Show that

$$E_0 - E_1 = -\int_N J^{\alpha}\chi_{\alpha}d^3x \tag{1}$$

where the integral is over the 3-dimensional portion of the horizon shown in the figure, and $\vec{\chi} \equiv \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$ is the normal to the horizon. Thus, $E_0 - E_1$, which is usually positive, equals the energy that flows down the horizon.

(c) The stress-energy tensor for a scalar field Φ is

$$T_{\mu\nu} = \Phi_{,\alpha}\Phi_{,\beta} - \frac{1}{2} g_{\alpha\beta}\Phi_{,\mu}\Phi^{,\mu}. \tag{2}$$

Show that if $\Phi = \Phi_0 \mathrm{Re}(e^{-i\omega t} e^{im\phi})$, where ω and m are constants, and if $\vec{\xi} = \partial/\partial t$, then

$$\langle -J^{\alpha}\chi_{\alpha} \rangle = \frac{1}{2}\Phi_0^2 \omega(\omega - m\Omega_H), \tag{3}$$

where $<>$ means a time average. Thus for $0 < \omega < m\Omega_H$, the energy falling into the black hole is *negative*, so the scalar field energy outside of the black hole is amplified over time.