Physics 236b assignment, Week 4:

(Jan 28, 2016. Due on Feb 4, 2016)

1. **Miller's Planet** [60 points]

   (*Don't be afraid to use tools like Mathematica for this problem.*)

   In the 2014 movie *Interstellar*, a planet orbits around a black hole, and is close enough to the black hole so that 1 hour spent on the planet equals 7 years at infinity.

   (a) Show that if the black hole is Schwarzschild, the orbit cannot be circular.

   (b) If the planet is in a stable circular equatorial orbit, what is the minimum possible value of the black hole spin? Hint: From your answer to part 1a, it should be clear that the spin needs to be quite close to extremal (and therefore the innermost stable orbit needs to be quite close to the horizon); therefore, it would be helpful to assume $a = M(1-\epsilon^3)$ for small ϵ and then solve for ϵ.

   (c) Show that the basis one-forms

$$\tilde{\omega}^{\hat{0}} = (\Delta/\Sigma)^{1/2}(dt - a\sin^2\theta d\phi), \tag{1}$$

$$\tilde{\omega}^{\hat{1}} = (\Delta/\Sigma)^{-1/2}dr, \tag{2}$$

$$\tilde{\omega}^{\hat{2}} = \Sigma^{1/2}d\theta, \tag{3}$$

$$\tilde{\omega}^{\hat{3}} = (\sin\theta/\Sigma^{1/2})\left(a\,dt - (r^2+a^2)d\phi\right), \tag{4}$$

   form an orthonormal basis for the Kerr metric. That is, the Kerr metric becomes $ds^2 = \eta_{\hat{\alpha}\hat{\beta}}\tilde{\omega}^{\hat{\alpha}}\tilde{\omega}^{\hat{\beta}}$.

   (d) Find $\vec{e}_{\hat{\alpha}}$, the orthonormal vector basis corresponding to $\tilde{\omega}^{\hat{\beta}}$, (i.e. $\langle\vec{e}_{\hat{\alpha}}, \tilde{\omega}^{\hat{\beta}}\rangle = \delta^{\hat{\beta}}_{\hat{\alpha}}$).

(e) If $\vec{u}$ is the 4-velocity of the planet, which is in a circular equatorial orbit about the black hole, show that

$$\vec{u} = (r^{-1}\Delta^{-1/2})\left(\tilde{E}(r^2+a^2) - a\tilde{L}\right)\vec{e}_{\hat{0}} + (1/r)(a\tilde{E} - \tilde{L})\vec{e}_{\hat{3}}, \tag{5}$$

where $\tilde{E}$ is the conserved energy per unit mass of the particle $-p_t/m$, and $\tilde{L}$ is the conserved angular momentum per unit mass of the particle p_ϕ/m.

(f) Consider the vectors

$$\vec{\lambda}_{\bar{0}} = \vec{u}, \tag{6}$$
$$\vec{\lambda}_{\bar{1}} = A\cos\Psi - B\sin\Psi, \tag{7}$$
$$\vec{\lambda}_{\bar{2}} = \vec{e}_{\hat{2}}, \tag{8}$$
$$\vec{\lambda}_{\bar{3}} = A\sin\Psi + B\cos\Psi, \tag{9}$$

where

$$A = \frac{\tilde{E}(r^2+a^2) - a\tilde{L}}{\sqrt{\Delta(r^2+K)}}\vec{e}_{\hat{1}}, \tag{10}$$
$$B = K^{1/2}\frac{\tilde{E}(r^2+a^2) - a\tilde{L}}{r\sqrt{\Delta(r^2+K)}}\vec{e}_{\hat{0}} + \frac{(r^2+K)^{1/2}(a\tilde{E}-\tilde{L})}{rK^{1/2}}\vec{e}_{\hat{3}}, \tag{11}$$
$$K = (a\tilde{E} - \tilde{L})^2, \tag{12}$$

and where Ψ is an angle.

Show that $(\vec{\lambda}_{\bar{0}}, \vec{\lambda}_{\bar{1}}, \vec{\lambda}_{\bar{2}}, \vec{\lambda}_{\bar{3}})$ also form an orthonormal basis. Note that $\vec{\lambda}_{\bar{1}}$ and $\vec{\lambda}_{\bar{3}}$ rotate in the "$r-\phi$" plane in the frame of the planet if the angle Ψ changes. Also (you don't need to show this; it is difficult and tedious) if the angle Ψ obeys

$$\frac{d\Psi}{d\tau} = (r^2+K)^{-1}K^{1/2}\left(\tilde{E} - a/(a\tilde{E}-\tilde{L})\right), \tag{13}$$

then it turns out that all of the $\vec{\lambda}_{\bar{\alpha}}$ are parallel transported along $\vec{u}$. Thus $(\vec{\lambda}_{\bar{0}}, \vec{\lambda}_{\bar{1}}, \vec{\lambda}_{\bar{2}}, \vec{\lambda}_{\bar{3}})$ corresponds to the local Lorentz frame of the planet.

Why do we care about the orthonormal basis $(\vec{\lambda}_{\bar{0}}, \vec{\lambda}_{\bar{1}}, \vec{\lambda}_{\bar{2}}, \vec{\lambda}_{\bar{3}})$? We care because it simplifies the next part of the problem. In J.-A. Marck, Proc. R. Soc. Lond. A 385, 431 (1983) it was shown that that the components of the Riemann tensor in the basis $(\vec{\lambda}_{\bar{0}}, \vec{\lambda}_{\bar{1}}, \vec{\lambda}_{\bar{2}}, \vec{\lambda}_{\bar{3}})$ take a particularly simple form:

$$R_{\bar{0}\bar{1}\bar{0}\bar{1}} = \left(1 - 3\frac{r^2+K}{r^2}\cos^2\Psi\right)\frac{M}{r^3}, \tag{14}$$

$$R_{\bar{0}\bar{2}\bar{0}\bar{2}} = \left(1 + 3\frac{K}{r^2}\right)\frac{M}{r^3}, \tag{15}$$

$$R_{\bar{0}\bar{3}\bar{0}\bar{3}} = \left(1 - 3\frac{r^2+K}{r^2}\sin^2\Psi\right)\frac{M}{r^3}, \tag{16}$$

$$R_{\bar{0}\bar{1}\bar{0}\bar{3}} = -3\left(\frac{r^2+K}{r^2}\right)\frac{M}{r^3}\sin\Psi\cos\Psi. \tag{17}$$

We will use this for the next part of the problem.

(g) Assume the planet has the same size and mass as Earth. Someone on the surface of the planet will feel the (Newtonian) gravitational acceleration $g = 9.8\,\mathrm{m/s^2}$ downward, but will also feel a tidal force from the black hole that (for some observers) will point upward. If this tidal force is greater than g, then the planet will be ripped apart. What is the minimum mass of the black hole such that the planet remains intact? Assume the basis and Riemann tensor from the previous part of the problem.