Physics 236b assignment, Week 3:

(Jan 21, 2016. Due on Jan 28, 2016)

1. **Particle orbits in Reissner-Nordström** [15 points]

   Consider a test particle with electric charge q and 4-momentum $\vec{p} = d/d\lambda$ moving in curved spacetime with metric $g_{\mu\nu}$. Assume there is also an electromagnetic field with vector potential A_μ. Because there is an electromagnetic force, the particle does not move along a geodesic, but instead has an acceleration given by the Lorentz force.

   We will show that the particle motion can be described by a Hamiltonian

   $$H = \frac{1}{2} g^{\mu\nu}(\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu), \tag{1}$$

   where A_μ and $g_{\mu\nu}$ are considered functions of the coordinates x^μ, and the Hamiltonian is considered a function of x^μ and some generalized momentum π_μ.

   (a) Hamilton's equations read

   $$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi_\mu}, \tag{2}$$

   $$\frac{d\pi_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}. \tag{3}$$

   From Eq. (2), find the 4-momentum p_μ in terms of the generalized momementum π_μ.

   (b) Write down the geodesic equation for p^μ, including the acceleration term due to the electromagnetic field.

   (c) Show that Eq. (3) reduces to the result of part (1b).

(d) Now assume that the metric and vector potential are given by the Reissner-Nordström solution for a black hole of mass M and charge Q. Also assume that the particle moves on an equatorial orbit, so $\theta = \pi/2$ and $p^\theta = 0$. Show that the two quantities

$$E = -p_t + q\frac{Q}{r}, \tag{4}$$

$$L = p_\phi, \tag{5}$$

are conserved along the path of the particle. (Hint: the Hamiltonian is independent of t and ϕ).

(e) Show that the motion of the particle in part (1d) can be written as a motion in an effective potential

$$(dr/d\lambda)^2 + V(r) = E^2, \tag{6}$$

where

$$V(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(m^2 + \frac{L^2}{r^2}\right) - \frac{Q^2q^2}{r^2} + \frac{2QqE}{r}, \tag{7}$$

and m is the rest mass of the particle.

2. **Observer on a collapsing star** [10 points] Consider on observer on the surface of a collapsing star.

(a) Show that if $\vec{p}$ is the 4-momentum of a photon, $\vec{u}$ is the 4-velocity of the observer on the surface of the star, and $\vec{n}$ is a unit 4-vector normal to the surface, then

$$\cos\theta = \frac{\vec{n}\cdot\vec{p}}{\vec{p}\cdot\vec{u}}, \tag{8}$$

where θ is the angle, measured by the observer, between the normal to the surface and the direction of the photon.

(b) Show that for spherical collapse,

$$\cos\theta = -\frac{\alpha\sqrt{1-\alpha b^2/r^2} - v_s}{\alpha - v_s\sqrt{1-\alpha b^2/r^2}}, \tag{9}$$

where $\alpha \equiv 1-2M/r$, $v_s = (dr/dt)_{\text{observer}}$, and b is the photon impact parameter $b \equiv L/E$.

(c) Evaluate part 2b for a photon emitted at $r = 3M$ into a circular orbit. Comment.

3. **Oppenheimer-Snyder Collapse and Apparent Horizons** [35 points]

Consider the collapse of a spherical dustball (zero pressure, uniform density perfect fluid) of mass M which begins at rest with finite Schwarzschild radius $R = R_i = 5M$ at time $t = 0$.

The metric is given by the Schwarzshild solution outside the matter ($r > R$),

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{10}$$

and the Friedmann solution inside the matter ($r < R$),

$$ds^2 = -d\tau^2 + a^2(\tau)\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right). \tag{11}$$

(a) Let r_i be the circumferential radius of a given mass element i in the dustball. Consider the elements $i = 0$, $1/4$, $1/2$, and 1, encompassing a fraction i of the total mass. Plot the worldlines $r_i(\tau)$ on a spacetime diagram, with proper time (i.e. synchronous coordinate time) along the vertical axis, and circumferential radius along the horizontal axis.

(b) Consider an outgoing spherical flash of light, i.e. a sphere of outgoing radial photons. As this flash propagates outward, the surface area of the sphere of photons changes. A *trapped surface* is a region of spacetime in which the surface area of such an outgoing sphere of photons is momentarily *decreasing*. An *apparent horizon* is the outermost trapped surface. Compute the radius of the apparent horizon, $r_{\rm AH}$, as a function of proper time τ and plot it on the spacetime diagram from part 3a.

(c) On the same spacetime diagram, plot the *event horizon*, $r_{\rm H}(\tau)$, which is the inner boundary of the region in which outgoing radial photons can escape to infinity.

(d) Tabulate your results as follows:

$\tau/\tau_{\rm max}$	r_i/M (for all i)	$r_{\rm AH}/M$	$r_{\rm H}/M$
0.0			
0.2			
0.4			
0.6			
0.8			
1.0			