Physics 236b assignment, Week 1:

(Jan 07, 2016. Due on Jan 14, 2016)

1. **Speed of sound** [15 points]

   A perfect fluid has stress-energy tensor

   $$T^{\alpha\beta} = (P + \rho)u^\alpha u^\beta + P g^{\alpha\beta}. \tag{1}$$

   Assume an isentropic perturbation $P = P_0 + \delta P$, $\rho = \rho_0 + \delta\rho$, $u^i = (1, v_i)$, where δP, $\delta\rho$, v_i are small. Work in the LLF of the fluid, so the metric is flat and the (unperturbed) fluid is at rest.

   (a) Use $T^{\alpha\beta}{}_{;\beta} = 0$ to derive evolution equations for the perturbed quantities.

   (b) For an isentropic perturbation we can assume

   $$\delta P = \left.\frac{\partial P}{\partial \rho}\right|_s \delta\rho, \tag{2}$$

   where $\left.\frac{\partial P}{\partial \rho}\right|_s$ can be considered a constant (since it is computed from the unperturbed fluid). Show that the perturbations propagate as waves (sound waves), with speed v_s, where

   $$v_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{3}$$

2. **Isotropic cordinates** [15 points]

   As done in class, spherical spacetimes are often expressed in terms of Schwarzshild coordinates,

   $$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2. \tag{4}$$

However, another useful set of coordinates is *isotropic coordinates*, in which the metric is

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\mu}(d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{5}$$

(a) Compute μ and $\bar{r}$ in terms of r and λ. (The answer may still have an integral in it).

(b) Assume vacuum, so that $e^{2\Phi} = e^{-2\lambda} = 1 - 2M/r$. Derive a closed-form expression for the Schwarzschild metric in isotropic coordinates.

(c) What is the area of a sphere with constant $\bar{r}$ and t?

(d) Draw an embedding diagram $z(r)$ for the spacelike hypersurface $t = 0$ of the Schwarzschild geometry at a fixed θ, ϕ. Indicate regions where $\bar{r} \to 0$ and where $\bar{r} \to \infty$.

3. **Fermion star** [20 points]

(This problem looks long, but most is explanation.)

In special relativistic kinetic theory, a system of particles is described by the phase space distribution function

$$\frac{d\mathcal{N}}{d^3x\, d^3p}. \tag{6}$$

Here $\frac{d\mathcal{N}}{d^3x\, d^3p}$ represents the number of particles per unit volume in phase space (here d^3p is the volume element in momentum space). Therefore the number density of particles n, the total energy density ρ, and the pressure P are given by

$$n = \int \frac{d\mathcal{N}}{d^3x\, d^3p} d^3p, \tag{7}$$

$$\rho = \int (p^2 + m^2)^{1/2} \frac{d\mathcal{N}}{d^3x\, d^3p} d^3p, \tag{8}$$

$$P = \frac{1}{3} \int pv \frac{d\mathcal{N}}{d^3x\, d^3p} d^3p. \tag{9}$$

Here $(p^2 + m^2)^{1/2}$ is the total (special relativistic) energy of one particle, and pv is the momentum flux of one particle (pressure is a momentum flux). The factor of $1/3$ in the last equation comes from isotropy: we assume an equal momentum flux in every direction.

For an ideal gas of Fermions at zero temperature, all momentum states are filled up to a value called the Fermi momentum p_F, and all states with momentum above p_F are unoccupied, and therefore

$$\frac{d\mathcal{N}}{d^3x\, d^3p} = \begin{cases} \frac{2}{h^3}, & |\vec{p}| \leq p_F \\ 0, & |\vec{p}| > p_F. \end{cases} \tag{10}$$

Here h is Planck's constant, and the factor of 2 comes from 2 spin states per Fermion. The approximation of zero temperature is accurate when $p_F \gg kT$ (as in the case of white dwarf stars).

(a) For a zero-temperature Fermi gas, write down integral expressions for ρ, n, and P.

(b) Assume (for this part and for subsequent parts of the problem) that the rest mass of the Fermions is much smaller than the Fermi momentum, so that the rest mass can be neglected. In this case, what is the equation of state $P(\rho)$?

(c) What is the speed of sound in this star? (see problem 1b)

(d) Show that the equations of stellar structure have a solution $m(r) = 3r/14$.

(e) Find $\rho(r)$, $P(r)$, and $n(r)$.

(f) Although $n(0)$ is infinite, show that the number of particles inside any radius r is finite.

(g) Consider the 3-dimensional metric at a constant t. Draw an embedding diagram $z(r)$ of this metric on a $\theta, \phi =$ const surface. The kink at $r = 0$ is called a *conical singularity.*

4. **Coordinate-free expression for** $m(r)$ [10 points]

For a spherical star with a metric given by Eqs. (4) and

$$e^{-2\lambda} = 1 - 2m(r)/r, \tag{11}$$

express the function $m(r)$ in a coordinate-invariant manner (i.e. in terms of scalars, vectors, etc). Hint: the surface area A of a sphere of coordinate radius r is a measurable scalar.