Semi-analytical modeling of compact-binary mergers

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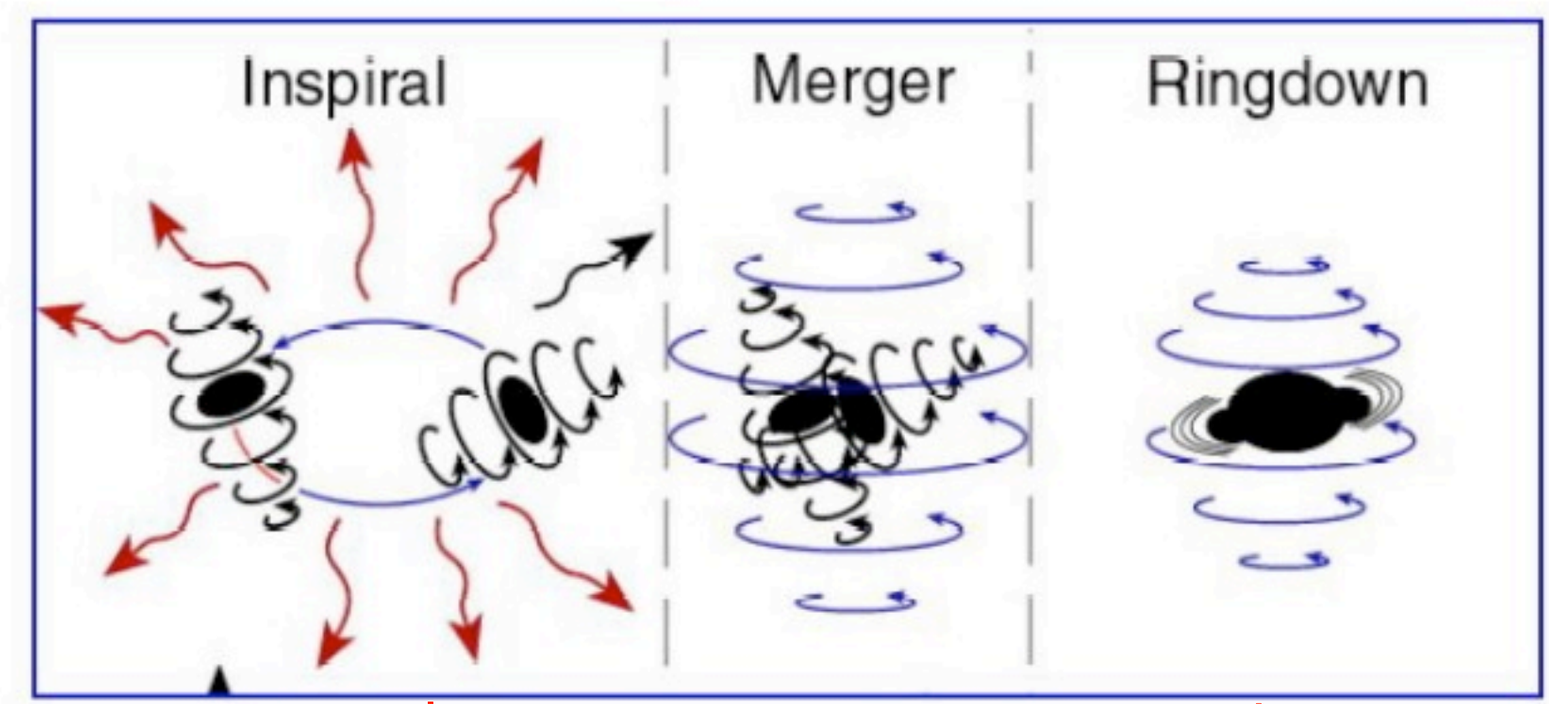
Overview

- Why semi-analytic methods?
  - Much faster than NR simulations
  - Provide physical intuitions/interpretations

- Treating BHs as two point particles
  - three stages of the coalescence
  - Post-Newtonian expansion
  - Black-hole perturbation theory

- Effect of spins

Three Stages of Binary Black Hole Coalescence



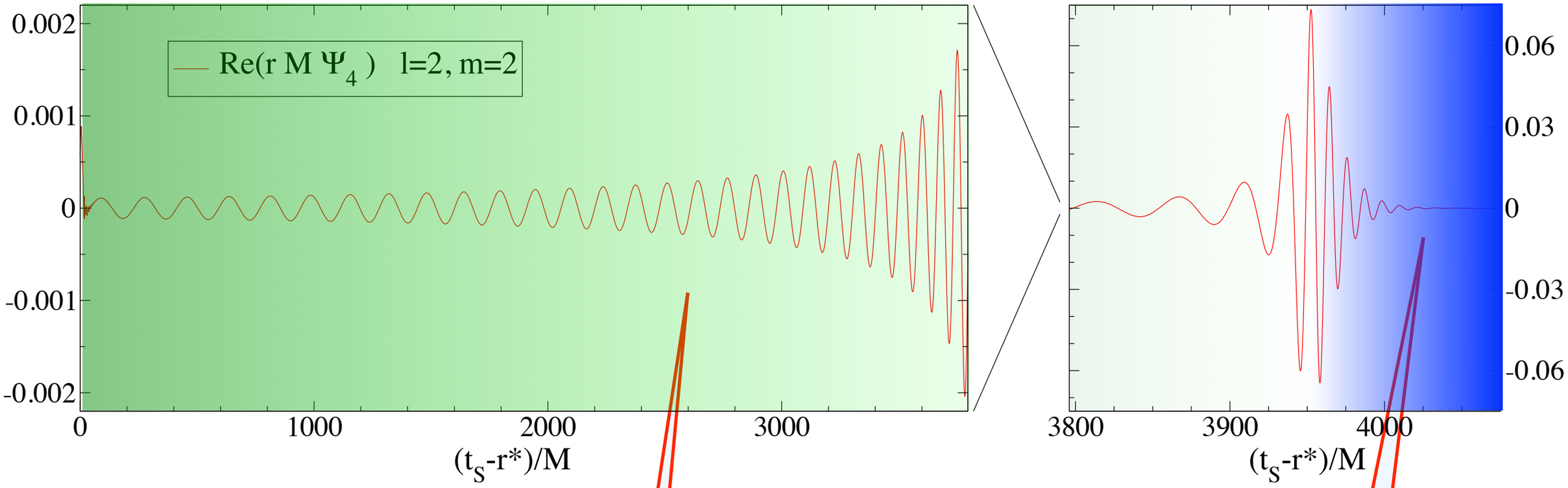
two nearly point particles encircling each other

"Post-Newtonian Expansion"

the final black hole settles down

Black-Hole Perturbation Theory

Three Stages of Binary Black Hole Coalescence



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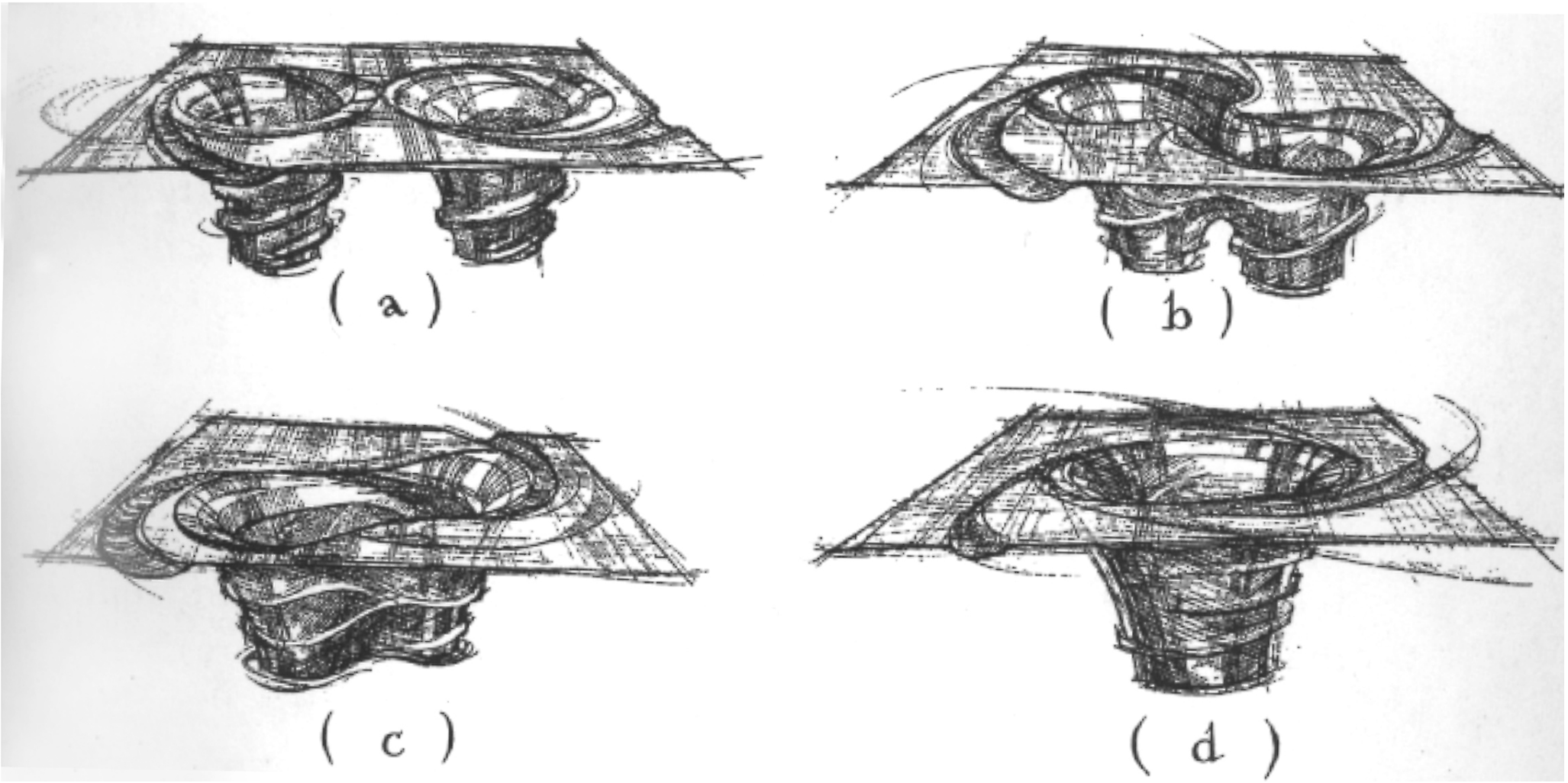
Black-Hole Perturbation Theory

“Post-Newtonian Expansion”



- In stage (a), treat the two black holes as two point particles
  - derive equations of motion *[requires regularization]*
  - compute gravitational wave forms
  - add “extended-object corrections”, e.g., spins, quadrupole moments, horizon

Einstein's Equation

$$G^{\alpha\beta}[g, \partial g, \partial^2 g] = \frac{8\pi G}{c^4} T^{\alpha\beta}[g]$$

Einstein's Equation

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$

Einstein Tensor

Bianchi Identity

$$\nabla_\alpha G^{\alpha\beta} = 0$$

$T^{\alpha\beta}$: stress-energy tensor

	0	1,2,3
0	*energy density*	*energy flux*
1,2,3	*momentum density*	*momentum flux (stress)*

$$\nabla_\alpha T^{\alpha\beta} = 0$$

energy-momentum conservation

Perturbation near Flat Spacetime

$$G^{\alpha\beta}[g, \partial g, \partial^2 g] = \frac{8\pi G}{c^4} T^{\alpha\beta}[g]$$

Einstein's Equation

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4} \tau^{\alpha\beta}$$

"wave equation"

$$h^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

deviation from flat spacetime

$$\partial_\mu h^{\alpha\mu} = 0$$

harmonic gauge

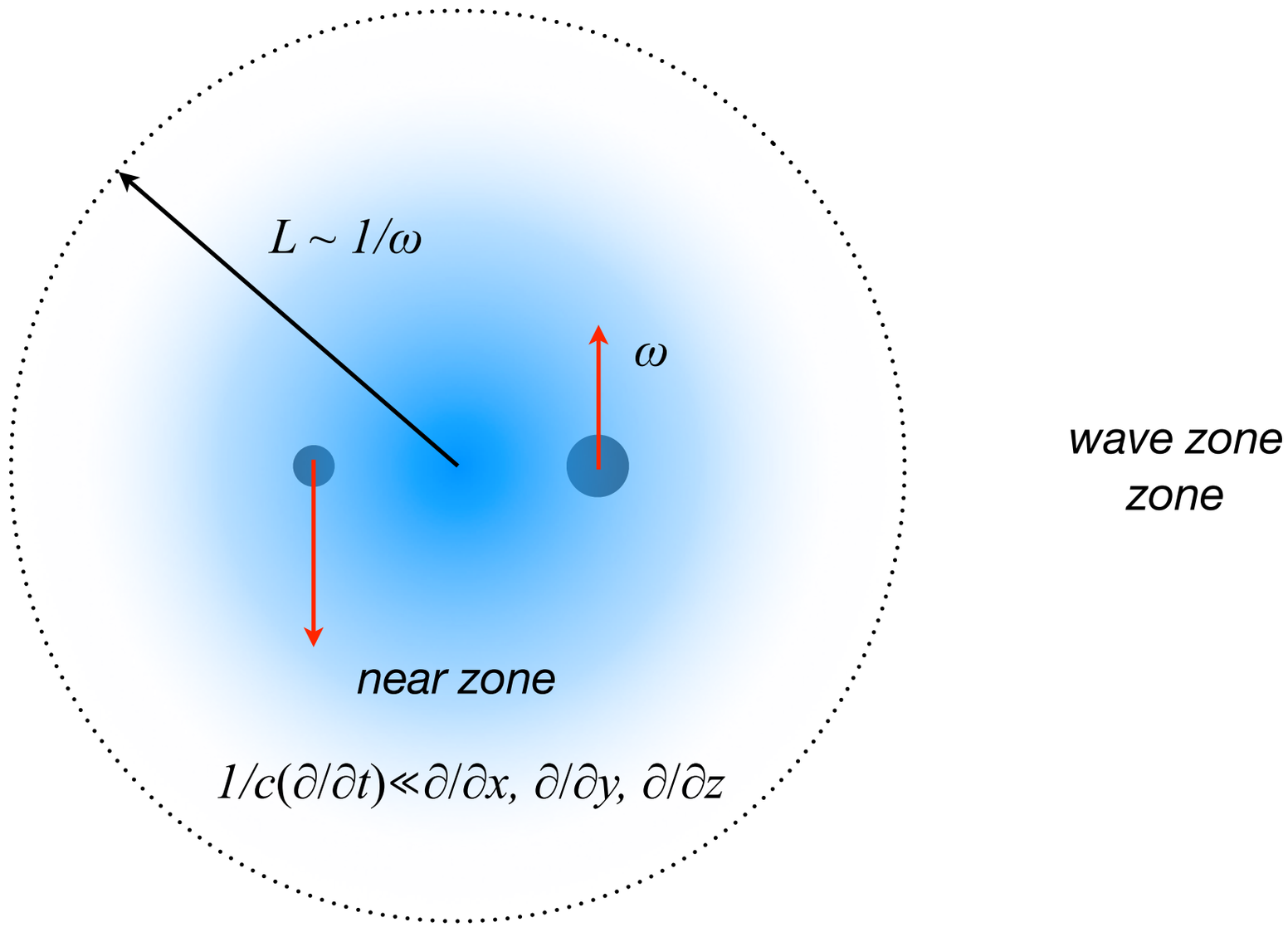
$$\tau^{\alpha\beta} = |g| T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta}$$

stress-energy tensor

nonlinear term

$$\begin{aligned}\Lambda^{\alpha\beta} = & -h^{\mu\nu}\partial^2_{\mu\nu} h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ & - g^{\alpha\mu} g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu} g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu} g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ & + \frac{1}{8}(2 g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu})(2 g_{\lambda\tau} g_{\epsilon\pi} - g_{\tau\epsilon} g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}.\end{aligned}$$

Near Zone vs. Wave Zone for Slow-Motion Source



- [from E&M Analogy]:
  - in the *near zone*, field nearly follows the motion of the particle
  - in the wave zone, field is self-sustained

Metric Perturbation in the Near Zone

$$-\frac{1}{c^2}\partial_t^2 h^{\alpha\beta} + \nabla^2 h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta} \longrightarrow \nabla^2 h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta} + \frac{1}{c^2}\partial_t^2 h^{\alpha\beta}$$

Wave Equation *Poisson Equation*

... iteratively obtain
metric perturbations

accuracy increase
by $1/c^2$
per iteration

Metric Perturbation in the Near Zone

$$\widehat{H}(\mathbf{q},\mathbf{p}) = \widehat{H}_{\rm Newt}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 1PN}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 2PN}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 3PN}(\mathbf{q},\mathbf{p}), \tag{56}$$

where

$$\widehat{H}_{\rm Newt}(\mathbf{q},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{q}, \tag{57}$$

$$\widehat{H}_{\rm 1PN}(\mathbf{q},\mathbf{p}) = \frac{1}{8}(3\eta-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left[(3+\eta)\mathbf{p}^2 + \eta(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{q} + \frac{1}{2q^2}, \tag{58}$$

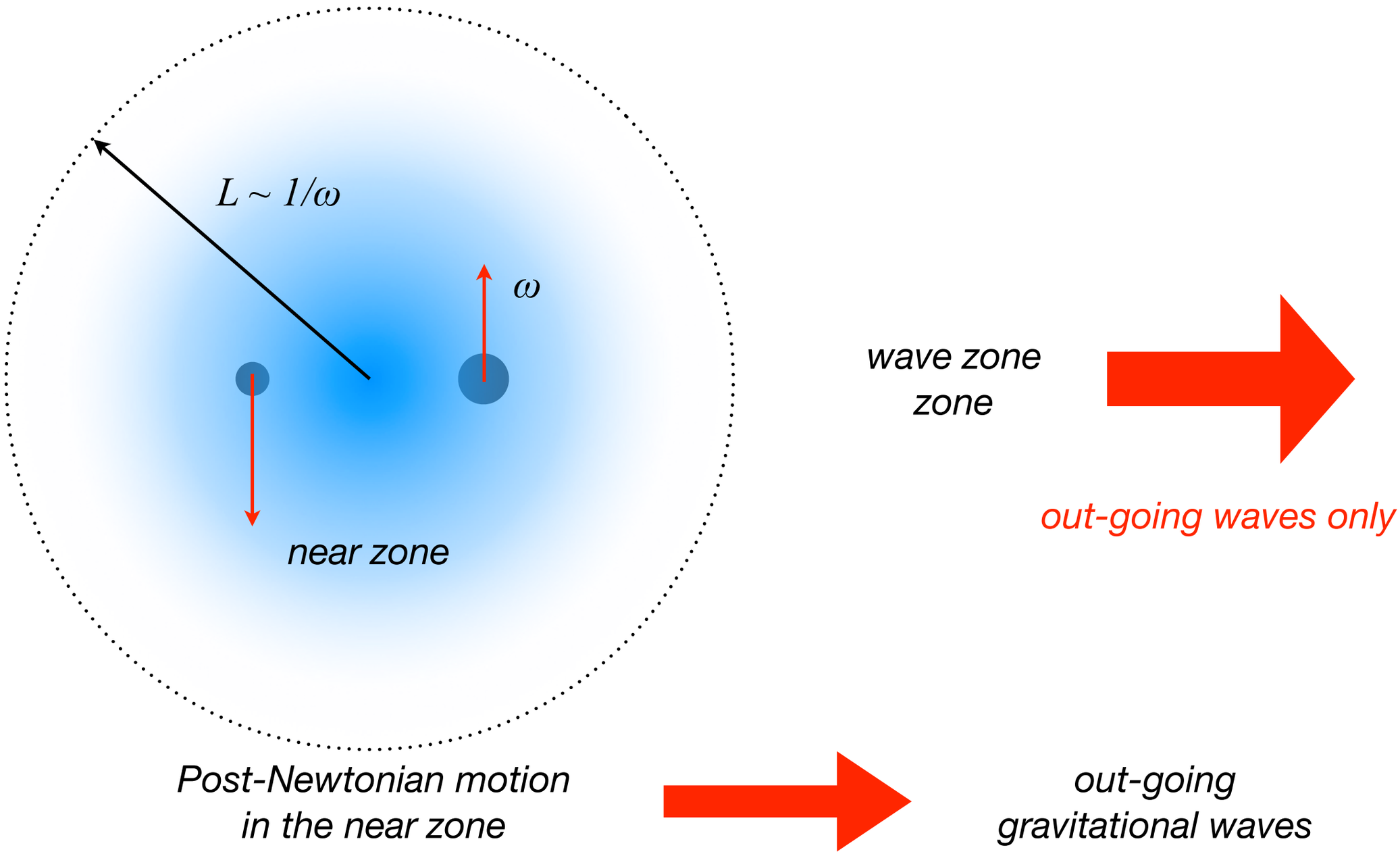
$$\begin{aligned}\widehat{H}_{\rm 2PN}(\mathbf{q},\mathbf{p}) = {} & \frac{1}{16}\left(1-5\eta+5\eta^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\left[\left(5-20\eta-3\eta^2\right)(\mathbf{p}^2)^2 - 2\eta^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\eta^2(\mathbf{n}\cdot\mathbf{p})^4\right]\frac{1}{q} \\ & + \frac{1}{2}\left[(5+8\eta)\mathbf{p}^2 + 3\eta(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{q^2} - \frac{1}{4}(1+3\eta)\frac{1}{q^3},\end{aligned} \tag{59}$$

$$\begin{aligned}\widehat{H}_{\rm 3PN}(\mathbf{q},\mathbf{p}) = {} & \frac{1}{128}\left(-5+35\eta-70\eta^2+35\eta^3\right)(\mathbf{p}^2)^4 \\ & + \frac{1}{16}\left[\left(-7+42\eta-53\eta^2-5\eta^3\right)(\mathbf{p}^2)^3 + (2-3\eta)\eta^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\eta)\eta^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\eta^3(\mathbf{n}\cdot\mathbf{p})^6\right]\frac{1}{q} \\ & + \left[\frac{1}{16}\left(-27+136\eta+109\eta^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\eta)\eta(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\eta)\eta(\mathbf{n}\cdot\mathbf{p})^4\right]\frac{1}{q^2} \\ & + \left\{\left[-\frac{25}{8} + \left(\frac{1}{64}\pi^2 - \frac{335}{48}\right)\eta - \frac{23}{8}\eta^2\right]\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3}{64}\pi^2 - \frac{7}{4}\eta\right)\eta(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{q^3} \\ & + \left[\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\eta\right]\frac{1}{q^4}.\end{aligned} \tag{60}$$

Hamiltonian for relative motion in the Center-of-Mass frame at 3PN or $1/c^6$

Radiation Reaction

- Break of time reversal symmetry given by out-going-wave boundary condition



Full Equations of Motion

- **Full equation of motion** derived up to 3.5 PN
  - conservative dynamics up to 3PN
  - radiation-reaction at 2.5PN and 3.5PN

$$\frac{dv^i}{dt} = -\frac{Gm}{r^2}\left[(1+\mathcal{A})\,n^i + \mathcal{B}\,v^i\right] + \mathcal{O}\left(\frac{1}{c^8}\right),$$

$$
\begin{aligned}
\mathcal{A} = \frac{1}{c^2}\Bigg\{ & -\frac{3\dot{r}^2\nu}{2} + v^2 + 3\nu v^2 - \frac{Gm}{r}(4+2\nu)\Bigg\} \\
& + \frac{1}{c^4}\Bigg\{ \frac{15\dot{r}^4\nu}{8} - \frac{45\dot{r}^4\nu^2}{8} - \frac{9\dot{r}^2\nu v^2}{2} + 6\dot{r}^2\nu^2 v^2 + 3\nu v^4 - 4\nu^2 v^4 \\
& \qquad + \frac{Gm}{r}\left(-2\dot{r}^2 - 25\dot{r}^2\nu - 2\dot{r}^2\nu^2 - \frac{13\nu v^2}{2} + 2\nu^2 v^2\right) + \frac{G^2m^2}{r^2}\left(9 + \frac{87\nu}{4}\right)\Bigg\} \\
& + \frac{1}{c^5}\left\{ -\frac{24\dot{r}\nu v^2}{5}\frac{Gm}{r} - \frac{136\dot{r}\nu}{15}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\Bigg\{ -\frac{35\dot{r}^6\nu}{16} + \frac{175\dot{r}^6\nu^2}{16} - \frac{175\dot{r}^6\nu^3}{16} + \frac{15\dot{r}^4\nu v^2}{2} - \frac{135\dot{r}^4\nu^2 v^2}{4} + \frac{255\dot{r}^4\nu^3 v^2}{8} \\
& \qquad - \frac{15\dot{r}^2\nu v^4}{2} + \frac{237\dot{r}^2\nu^2 v^4}{8} - \frac{45\dot{r}^2\nu^3 v^4}{2} + \frac{11\nu v^6}{4} - \frac{49\nu^2 v^6}{4} + 13\nu^3 v^6 \\
& \qquad + \frac{Gm}{r}\Bigg(79\dot{r}^4\nu - \frac{69\dot{r}^4\nu^2}{2} - 30\dot{r}^4\nu^3 - 121\dot{r}^2\nu v^2 + 16\dot{r}^2\nu^2 v^2 + 20\dot{r}^2\nu^3 v^2 + \frac{75\nu v^4}{4} \\
& \qquad\qquad + 8\nu^2 v^4 - 10\nu^3 v^4\Bigg) \\
& \qquad + \frac{G^2m^2}{r^2}\Bigg(\dot{r}^2 + \frac{32573\dot{r}^2\nu}{168} + \frac{11\dot{r}^2\nu^2}{8} - 7\dot{r}^2\nu^3 + \frac{615\dot{r}^2\nu\pi^2}{64} - \frac{26987\nu v^2}{840} + \nu^3 v^2 \\
& \qquad\qquad - \frac{123\nu\pi^2 v^2}{64} - 110\dot{r}^2\nu\ln\left(\frac{r}{r_0'}\right) + 22\nu v^2\ln\left(\frac{r}{r_0'}\right)\Bigg) \\
& \qquad + \frac{G^3m^3}{r^3}\left(-16 - \frac{437\nu}{4} - \frac{71\nu^2}{2} + \frac{41\nu\pi^2}{16}\right)\Bigg\} \\
& + \frac{1}{c^7}\Bigg\{ \frac{Gm}{r}\left(\frac{366}{35}\nu v^4 + 12\nu^2 v^4 - 114 v^2\nu\dot{r}^2 - 12\nu^2 v^2\dot{r}^2 + 112\nu\dot{r}^4\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(\frac{692}{35}\nu v^2 - \frac{724}{15}v^2\nu^2 + \frac{294}{5}\nu\dot{r}^2 + \frac{376}{5}\nu^2\dot{r}^2\right) \\
& \qquad + \frac{G^3m^3}{r^3}\left(\frac{3956}{35}\nu + \frac{184}{5}\nu^2\right)\Bigg\},
\end{aligned}
\tag{182}
$$

see, e.g., Blanchet, Living Reviews of Relativity, 9 (2006) 4

$$
\begin{aligned}
\mathcal{B} = {} & \frac{1}{c^2}\left\{-4\dot{r} + 2\dot{r}\nu\right\} \\
& + \frac{1}{c^4}\left\{\frac{9\dot{r}^3\nu}{2} + 3\dot{r}^3\nu^2 - \frac{15\dot{r}\nu v^2}{2} - 2\dot{r}\nu^2 v^2 + \frac{Gm}{r}\left(2\dot{r} + \frac{41\dot{r}\nu}{2} + 4\dot{r}\nu^2\right)\right\} \\
& + \frac{1}{c^5}\left\{\frac{8\nu v^2}{5}\frac{Gm}{r} + \frac{24\nu}{5}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{45\dot{r}^5\nu}{8} + 15\dot{r}^5\nu^2 + \frac{15\dot{r}^5\nu^3}{4} + 12\dot{r}^3\nu v^2 - \frac{111\dot{r}^3\nu^2 v^2}{4} - 12\dot{r}^3\nu^3 v^2 - \frac{65\dot{r}\nu v^4}{8}\right. \\
& \qquad + 19\dot{r}\nu^2 v^4 + 6\dot{r}\nu^3 v^4 \\
& \qquad + \frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6} + \frac{59\dot{r}^3\nu^2}{2} + 18\dot{r}^3\nu^3 - 15\dot{r}\nu v^2 - 27\dot{r}\nu^2 v^2 - 10\dot{r}\nu^3 v^2\right) \\
& \qquad \left. + \frac{G^2m^2}{r^2}\left(-4\dot{r} - \frac{18169\dot{r}\nu}{840} + 25\dot{r}\nu^2 + 8\dot{r}\nu^3 - \frac{123\dot{r}\nu\pi^2}{32} + 44\dot{r}\nu\ln\left(\frac{r}{r_0'}\right)\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\left(-\frac{626}{35}\nu v^4 - \frac{12}{5}\nu^2 v^4 + \frac{678}{5}\nu v^2\dot{r}^2 + \frac{12}{5}\nu^2 v^2\dot{r}^2 - 120\nu\dot{r}^4\right)\right. \\
& \qquad + \frac{G^2m^2}{r^2}\left(\frac{164}{21}\nu v^2 + \frac{148}{5}\nu^2 v^2 - \frac{82}{3}\nu\dot{r}^2 - \frac{848}{15}\nu^2\dot{r}^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-\frac{1060}{21}\nu - \frac{104}{5}\nu^2\right)\right\}.
\end{aligned}
$$

see, e.g., Blanchet, Living Reviews of Relativity, 9 (2006) 4

Adiabatic Evolution of Circular Orbits

- Energy/Angular momentum flux derived up to 3.5 Post leading order for circular orbits.

$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

$$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3} = (v/c)^2$$

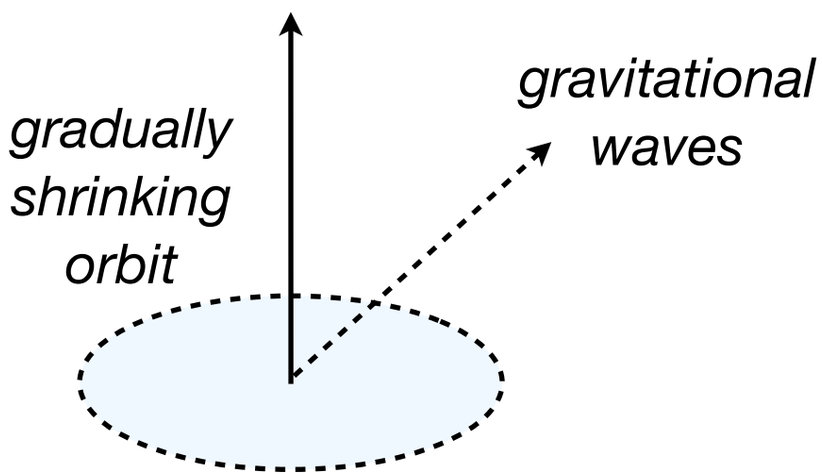
expansion parameter

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}.$$

Adiabatic Evolution: $\frac{dE}{dt} = \mathcal{L} \Rightarrow \dot{x} = \frac{\mathcal{L}}{dE/dx}$

Waveforms

- Having $x(t)=[v(t)/c]^2$, one can construct a series of circular orbits evolving in time
- One can also obtain *multipole radiation*



$$h_{+,\times} = \frac{2G\mu x}{c^2 R}\left\{H^{(0)}_{+,\times} + x^{1/2}H^{(1/2)}_{+,\times} + xH^{(1)}_{+,\times} + x^{3/2}H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + x^{5/2}H^{(5/2)}_{+,\times} + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}. \tag{238}$$

quadrupole waveform *(l=2)* at leading order

higher multipole waves *(l>2)* and corrections to the quadrupole wave

$$H_+^{(0)} = -(1+c_i^2)\cos 2\psi,$$

ψ: orbital phase

leading quadrupole contribution

$$H_+^{(1/2)} = -\frac{s_i}{8}\frac{\delta m}{m}\left[(5+c_i^2)\cos\psi - 9(1+c_i^2)\cos 3\psi\right],$$

$$H_+^{(1)} = \frac{1}{6}\left[19+9c_i^2-2c_i^4-\nu(19-11c_i^2-6c_i^4)\right]\cos 2\psi - \frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\cos 4\psi,$$

$$\begin{aligned} H_+^{(3/2)} = \frac{s_i}{192}\frac{\delta m}{m}\Big\{&\left[57+60c_i^2-c_i^4-2\nu(49-12c_i^2-c_i^4)\right]\cos\psi \\ &-\frac{27}{2}\left[73+40c_i^2-9c_i^4-2\nu(25-8c_i^2-9c_i^4)\right]\cos 3\psi \\ &+\frac{625}{2}(1-2\nu)s_i^2(1+c_i^2)\cos 5\psi\Big\} - 2\pi(1+c_i^2)\cos 2\psi, \end{aligned}$$

$$\begin{aligned} H_+^{(2)} = \frac{1}{120}\Big[&22+396c_i^2+145c_i^4-5c_i^6+\frac{5}{3}\nu(706-216c_i^2-251c_i^4+15c_i^6) \\ &-5\nu^2(98-108c_i^2+7c_i^4+5c_i^6)\Big]\cos 2\psi \\ &+\frac{2}{15}s_i^2\left[59+35c_i^2-8c_i^4-\frac{5}{3}\nu(131+59c_i^2-24c_i^4)+5\nu^2(21-3c_i^2-8c_i^4)\right]\cos 4\psi \\ &-\frac{81}{40}(1-5\nu+5\nu^2)s_i^4(1+c_i^2)\cos 6\psi \\ &+\frac{s_i}{40}\frac{\delta m}{m}\Big\{\left[11+7c_i^2+10(5+c_i^2)\ln 2\right]\sin\psi - 5\pi(5+c_i^2)\cos\psi \\ &-27\left[7-10\ln(3/2)\right](1+c_i^2)\sin 3\psi + 135\pi(1+c_i^2)\cos 3\psi\Big\}. \end{aligned}$$

see, e.g., Blanchet, Living Reviews of Relativity, 9 (2006) 4

$$H_\times^{(0)} = -2c_i \sin 2\psi,$$

ψ: orbital phase

leading quadrupole contribution

$$H_\times^{(1/2)} = -\frac{3}{4} s_i c_i \frac{\delta m}{m} \left[\sin\psi - 3\sin 3\psi\right],$$

$$H_\times^{(1)} = \frac{c_i}{3}\left[17 - 4c_i^2 - \nu(13 - 12c_i^2)\right]\sin 2\psi - \frac{8}{3}(1-3\nu)c_i s_i^2 \sin 4\psi,$$

$$H_\times^{(3/2)} = \frac{s_i c_i}{96}\frac{\delta m}{m}\Big\{\left[63 - 5c_i^2 - 2\nu(23 - 5c_i^2)\right]\sin\psi - \frac{27}{2}\left[67 - 15c_i^2 - 2\nu(19 - 15c_i^2)\right]\sin 3\psi$$
$$+\frac{625}{2}(1-2\nu)s_i^2 \sin 5\psi\Big\} - 4\pi c_i \sin 2\psi,$$

$$H_\times^{(2)} = \frac{c_i}{60}\left[68 + 226c_i^2 - 15c_i^4 + \frac{5}{3}\nu(572 - 490c_i^2 + 45c_i^4) - 5\nu^2(56 - 70c_i^2 + 15c_i^4)\right]\sin 2\psi$$
$$+\frac{4}{15}c_i s_i^2\left[55 - 12c_i^2 - \frac{5}{3}\nu(119 - 36c_i^2) + 5\nu^2(17 - 12c_i^2)\right]\sin 4\psi$$
$$-\frac{81}{20}(1 - 5\nu + 5\nu^2)c_i s_i^4 \sin 6\psi$$
$$-\frac{3}{20}s_i c_i \frac{\delta m}{m}\left\{[3 + 10\ln 2]\cos\psi + 5\pi\sin\psi - 9\left[7 - 10\ln(3/2)\right]\cos 3\psi - 45\pi\sin 3\psi\right\}.$$

see, e.g., Blanchet, Living Reviews of Relativity, 9 (2006) 4

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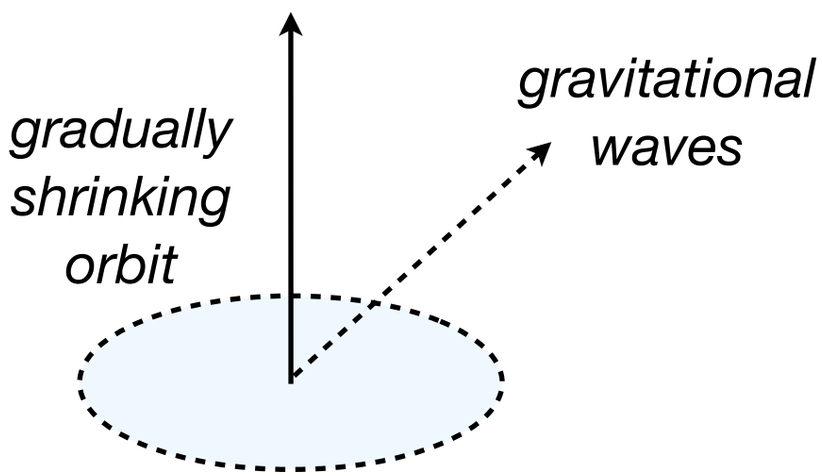

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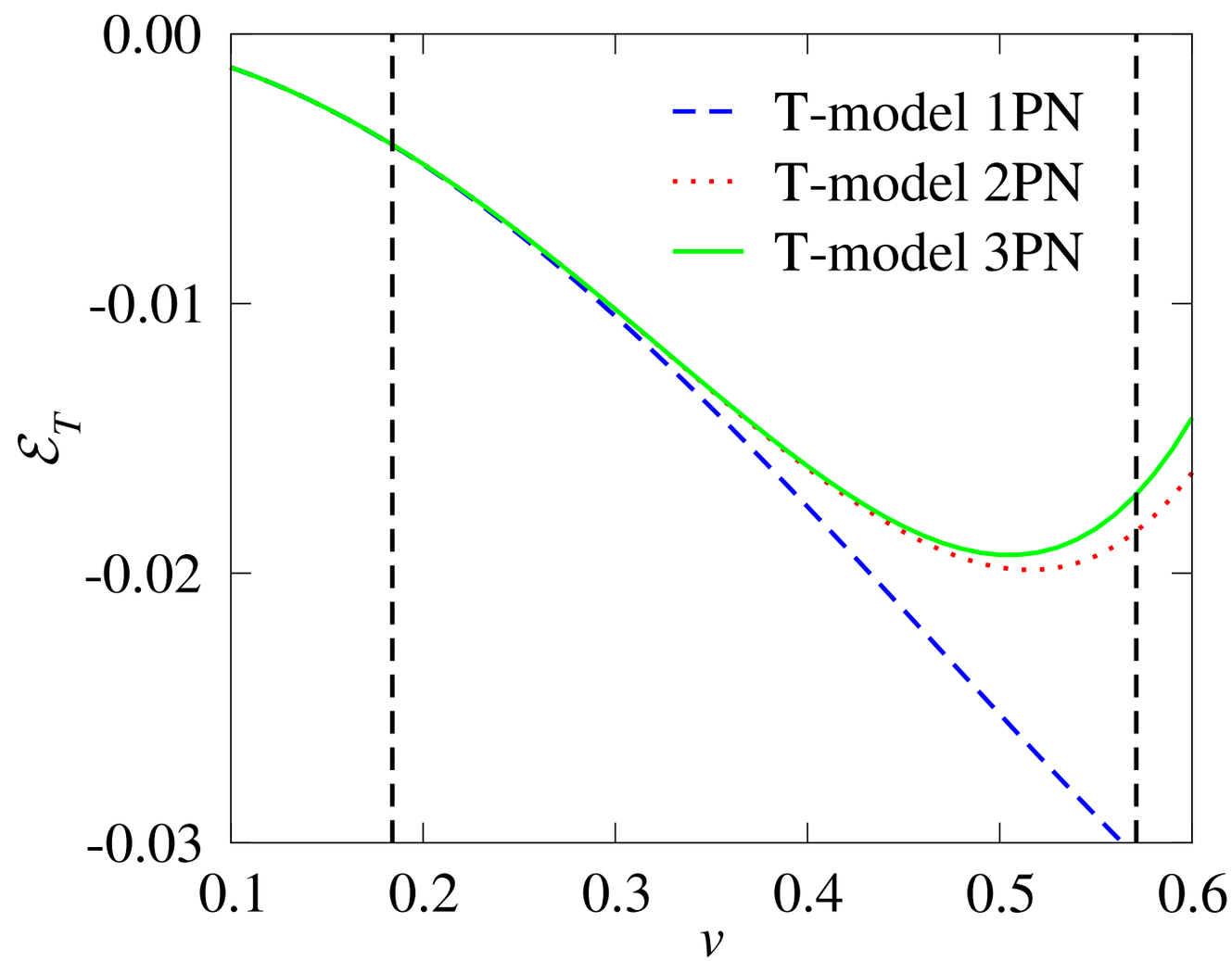
quadrupole waveform *(l=2)* at leading order

higher multipole waves *(l>2)* and corrections to the quadrupole wave

$$h_+ - ih_\times = \frac{1}{R}\sum_{l=2}^{+\infty}\sum_{m=-l}^{+l} H_{l,m}(t)\,{}_{-2}Y_{lm}(\theta,\varphi)$$

Ending of Post-Newtonian Regime?

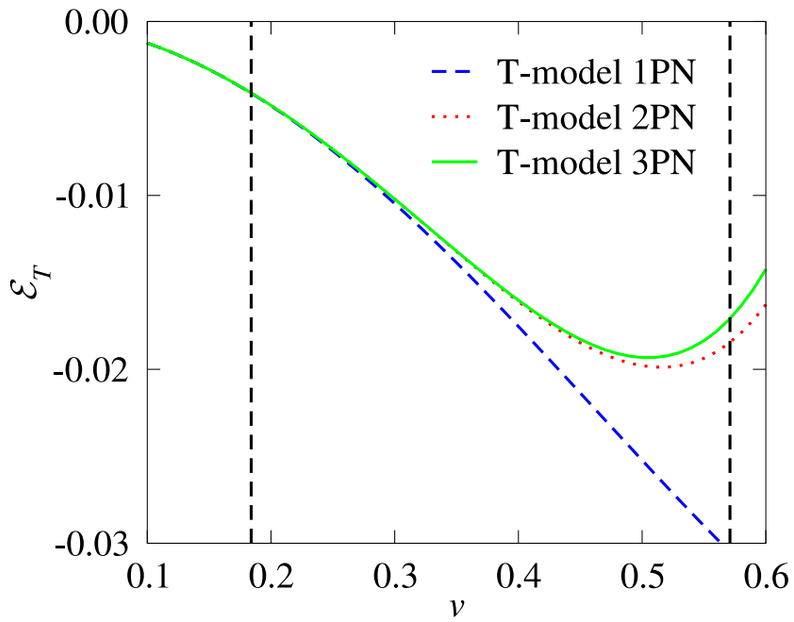
$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

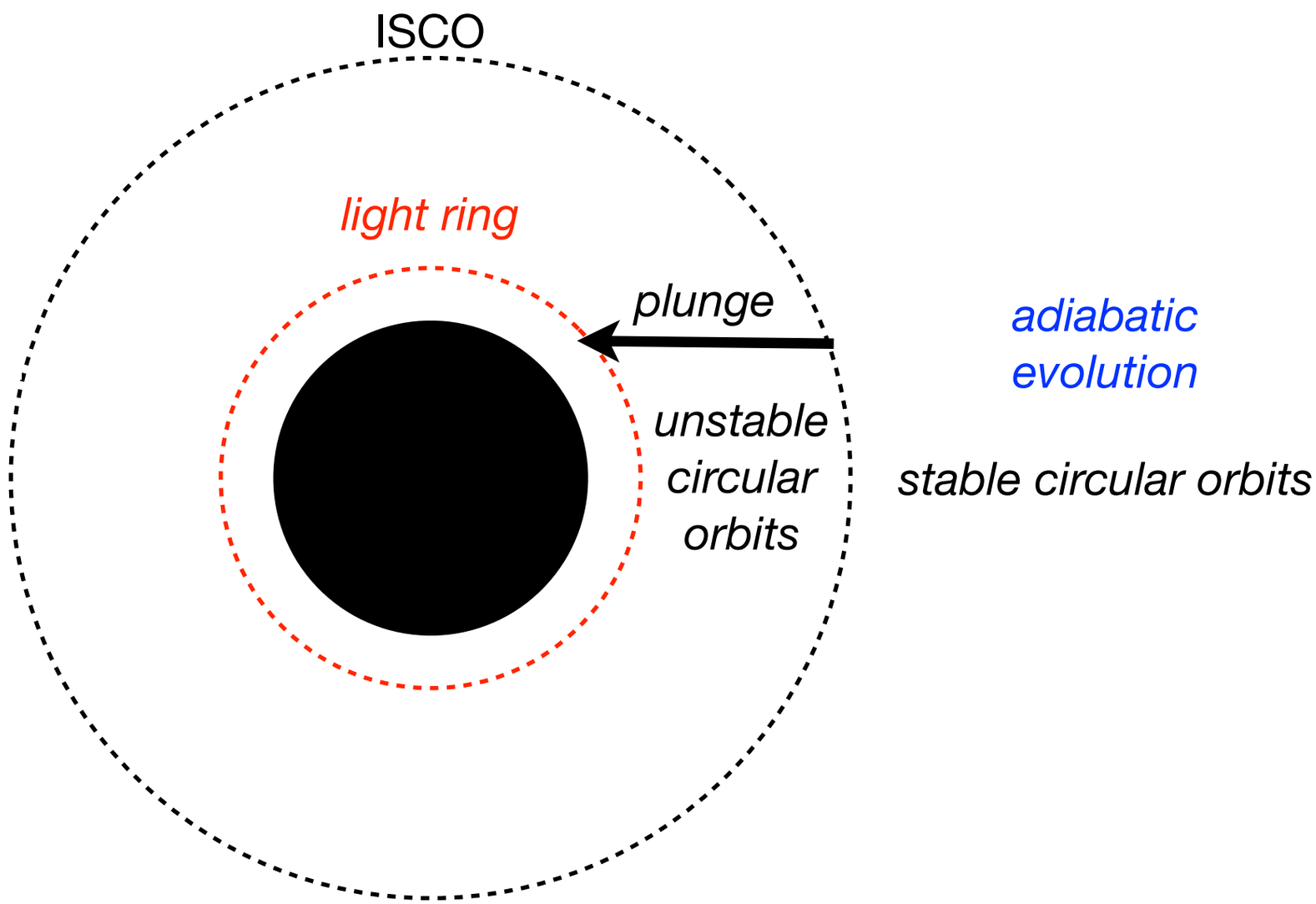


minimum energy for circular orbits [innermost stable circular orbit, or ISCO]

This is analogous to single BH ....

Single Particle Orbiting BH





Resummation Techniques

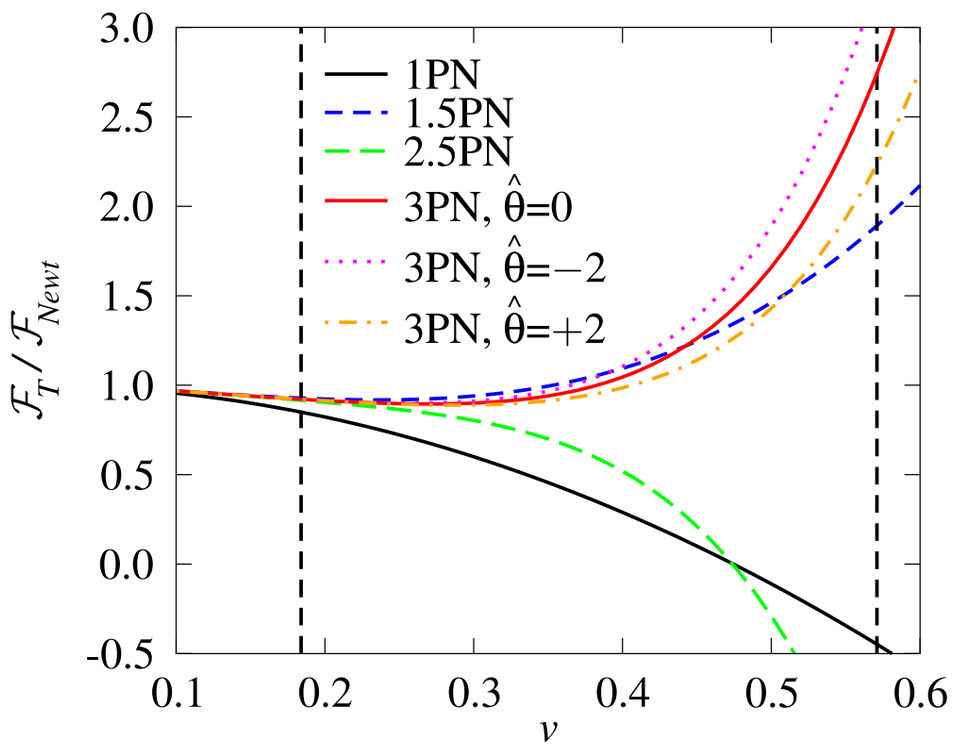
- Taylor expansion

$$y = \sum_n a_n x^n$$

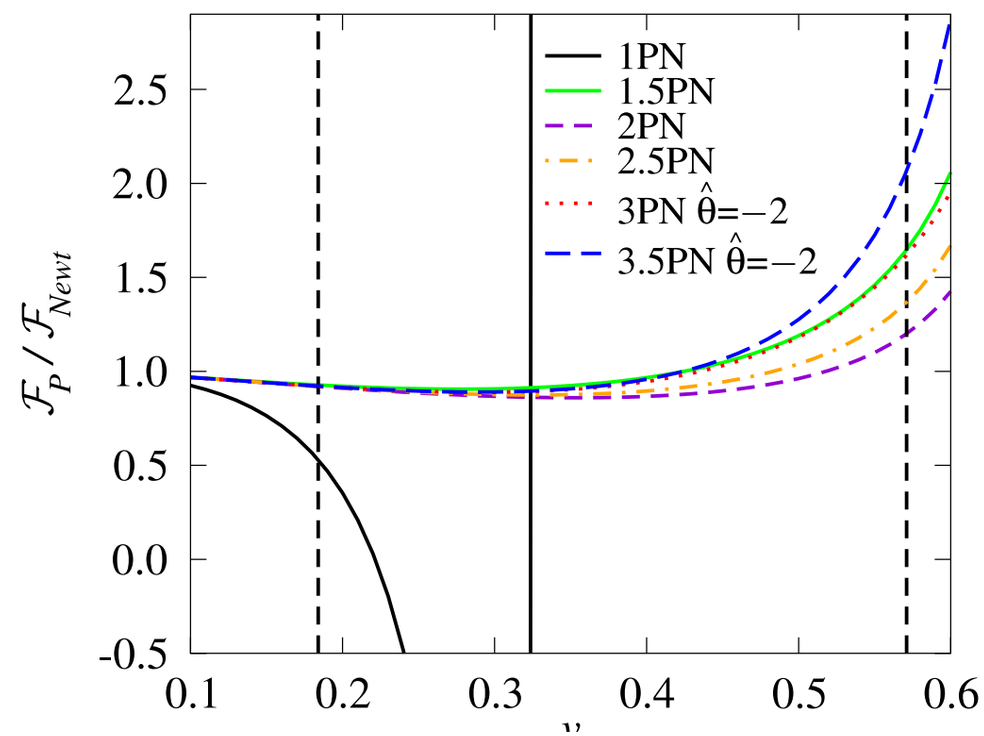
- Padé Approximant

$$y = \frac{\sum_n a_n x^n}{\sum_m b_m x^m}$$

- Example: flux



Taylor Expansions



Padé Approximants

"Variations on the Adiabatic PN evolution"

$$\frac{dx}{dt} = \frac{[\mathcal{L}(x)]_{\rm Taylor}}{[dE(x)/dx]_{\rm Taylor}}$$

Variation 1

$$\frac{dx}{dt} = \frac{[\mathcal{L}(x)]_{\rm Padé}}{[dE(x)/dx]_{\rm Padé}}$$

Variation 2

$$\left[\frac{dE/dx}{\mathcal{L}}\right]_{\rm Taylor} dx = dt \Rightarrow x^{\alpha}(1 \,\&\, x \,\&\, x^2 \,\&\, \ldots) = t$$

Variation 3

$$x = t^{1/\alpha}(1 \,\&\, t^{1/\alpha} \,\&\, t^{2/\alpha} \,\&\, \ldots)$$

Variation 4

... first obtain waveform, Fourier transform

$$h(f) = f^{-7/6} e^{i f^{-5/3}(1 \,\&\, f^{2/3} \,\&\, \ldots)}$$

Effective One Body Approach

- Instead of using Taylor-expanded Hamiltonian *[which doesn't recover ISCO...]*

$$\widehat{H}(\mathbf{q},\mathbf{p}) = \widehat{H}_{\rm Newt}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 1PN}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 2PN}(\mathbf{q},\mathbf{p}) + \widehat{H}_{\rm 3PN}(\mathbf{q},\mathbf{p}), \tag{56}$$

where

$$\widehat{H}_{\rm Newt}(\mathbf{q},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{q}, \tag{57}$$

$$\widehat{H}_{\rm 1PN}(\mathbf{q},\mathbf{p}) = \frac{1}{8}(3\eta-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left[(3+\eta)\mathbf{p}^2 + \eta(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{q} + \frac{1}{2q^2}, \tag{58}$$

$$\begin{aligned}\widehat{H}_{\rm 2PN}(\mathbf{q},\mathbf{p}) = {} & \frac{1}{16}\left(1-5\eta+5\eta^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\left[\left(5-20\eta-3\eta^2\right)(\mathbf{p}^2)^2 - 2\eta^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\eta^2(\mathbf{n}\cdot\mathbf{p})^4\right]\frac{1}{q} \\ & + \frac{1}{2}\left[(5+8\eta)\mathbf{p}^2 + 3\eta(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{q^2} - \frac{1}{4}(1+3\eta)\frac{1}{q^3},\end{aligned} \tag{59}$$

$$\begin{aligned}\widehat{H}_{\rm 3PN}(\mathbf{q},\mathbf{p}) = {} & \frac{1}{128}\left(-5+35\eta-70\eta^2+35\eta^3\right)(\mathbf{p}^2)^4 \\ & + \frac{1}{16}\left[\left(-7+42\eta-53\eta^2-5\eta^3\right)(\mathbf{p}^2)^3 + (2-3\eta)\eta^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\eta)\eta^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\eta^3(\mathbf{n}\cdot\mathbf{p})^6\right]\frac{1}{q} \\ & + \left[\frac{1}{16}\left(-27+136\eta+109\eta^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\eta)\eta(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\eta)\eta(\mathbf{n}\cdot\mathbf{p})^4\right]\frac{1}{q^2} \\ & + \left\{\left[-\frac{25}{8} + \left(\frac{1}{64}\pi^2 - \frac{335}{48}\right)\eta - \frac{23}{8}\eta^2\right]\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3}{64}\pi^2 - \frac{7}{4}\eta\right)\eta(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{q^3} \\ & + \left[\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\eta\right]\frac{1}{q^4}.\end{aligned} \tag{60}$$

consider a deformed spherically symmetric spacetime

Effective One Body Approach

- Deformed spherically symmetric spacetime

$$ds^2_{\rm eff} \equiv g^{\rm eff}_{\mu\nu}\, dx^\mu\, dx^\nu = -A(R)\, c^2 dt^2 + \frac{D(R)}{A(R)}\, dR^2 + R^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)\,,$$

$$\begin{aligned} A(R) &= 1 + a_1 \frac{GM}{c^2 R} + a_2 \left(\frac{GM}{c^2 R}\right)^2 + a_3 \left(\frac{GM}{c^2 R}\right)^3 + a_4 \left(\frac{GM}{c^2 R}\right)^4 + \cdots\,, \\ D(R) &= 1 + d_1 \frac{GM}{c^2 R} + d_2 \left(\frac{GM}{c^2 R}\right)^2 + d_3 \left(\frac{GM}{c^2 R}\right)^3 + \cdots\,. \end{aligned}$$

- Particle moving along some curve

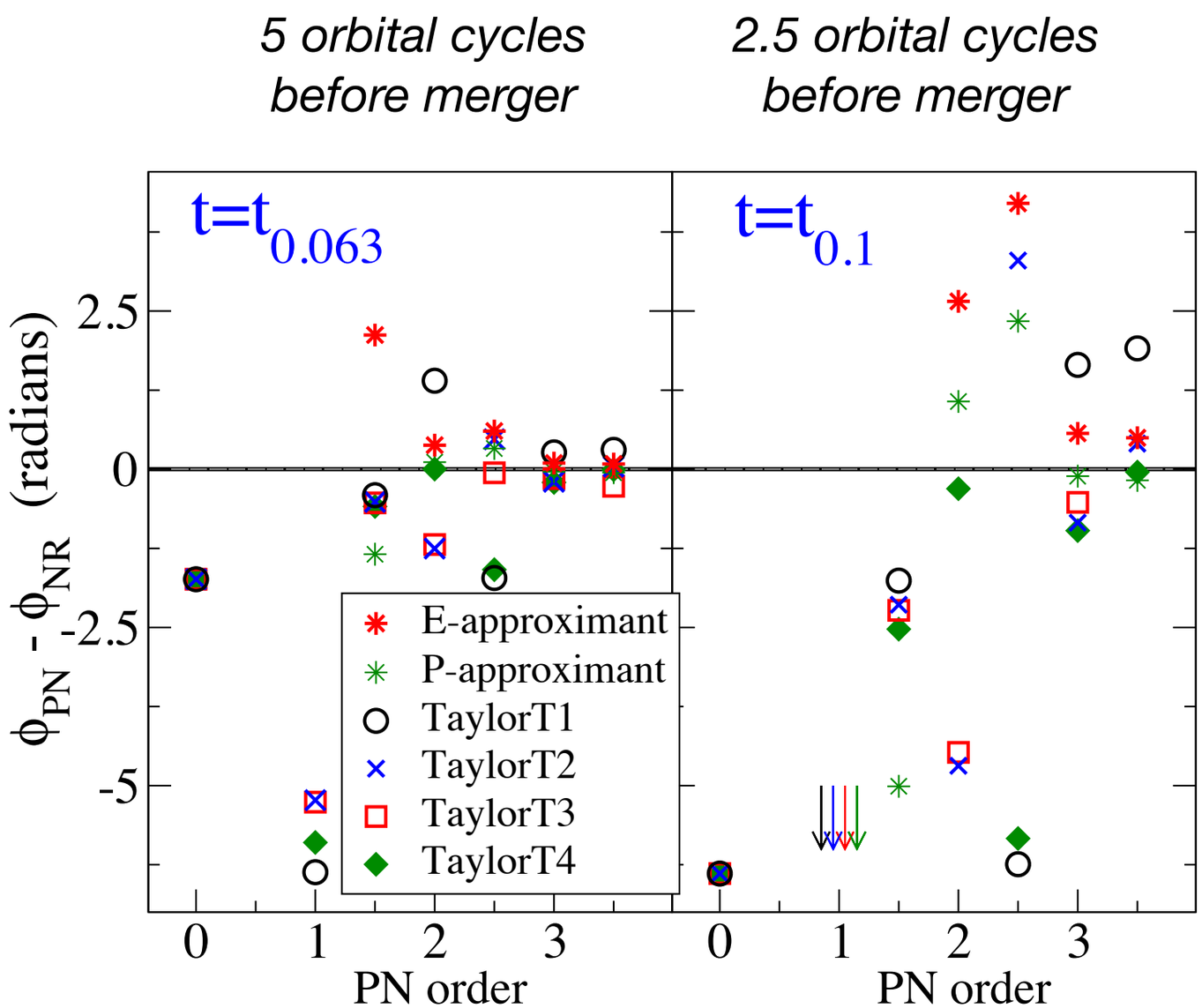
$$0 = \mu^2 + g^{\mu\eta}_{\rm eff}(x)\, p_\mu\, p_\eta + A^{\mu\eta\rho\sigma}(x)\, p_\mu\, p_\eta\, p_\rho\, p_\sigma + \cdots\,.$$

- Add *generalized force* that “implements” energy and angular momentum loss for circular orbits.

Original, Taylor coefficients in the PN expansion replaced by
$a_{1,2,3,\ldots}$, $d_{1,2,3,\ldots}$, and A

This allows the dynamics to have an ISCO at the end ...

Comparison of PN variations with Numerical Results



[Boyle et al., 2008]

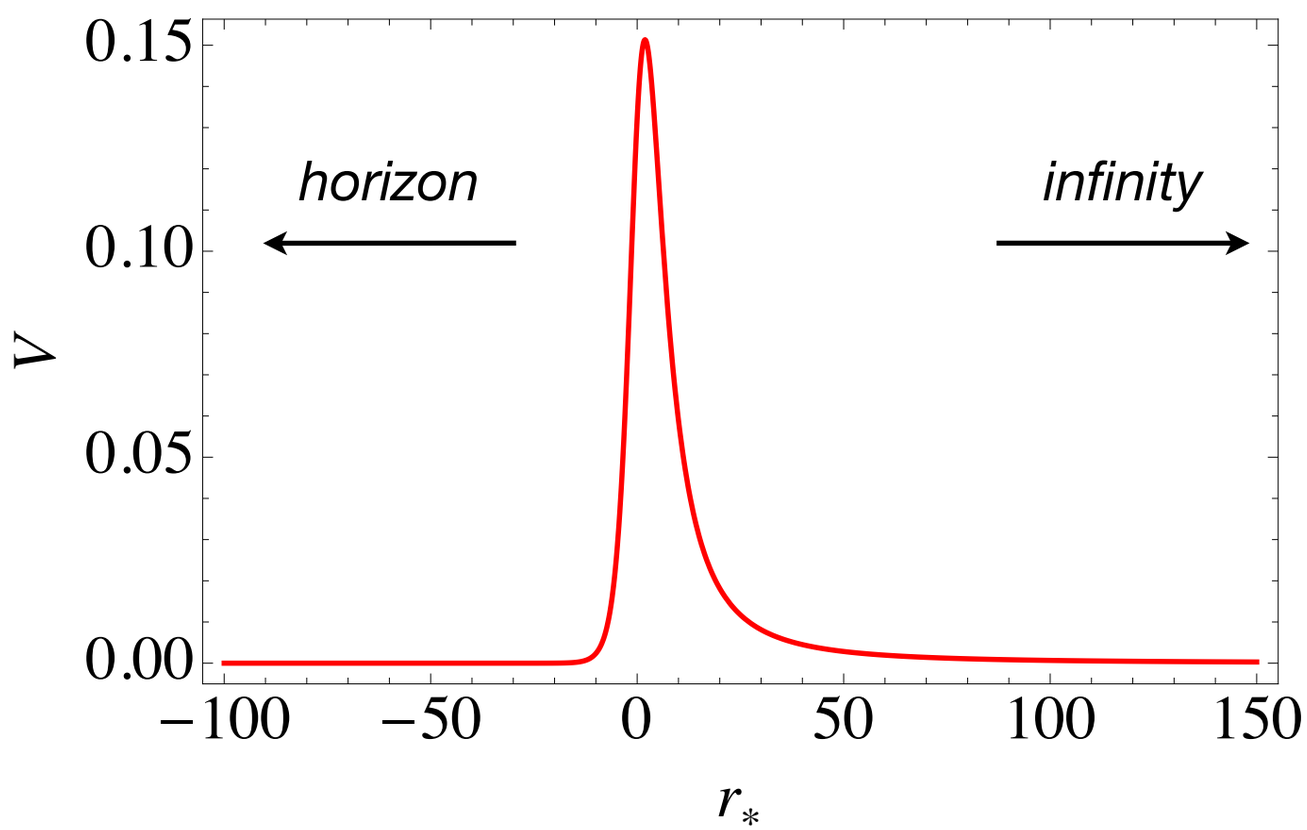
non-spinning binaries: post-Newtonian expansion can be used to approximate waveforms up to a few cycles before "merger"

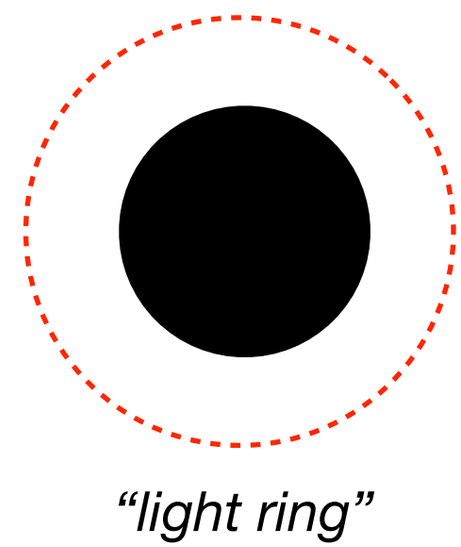
Black Hole Perturbation Theory

- Perturbations to the space-time geometry of a black hole are like waves propagating around.

$$\nabla^\alpha \nabla_\alpha \bar{h}_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta \bar{h}_{\alpha\beta} = 0 .$$

- In the radial direction, use r* coordinate, and we have, for a particular l

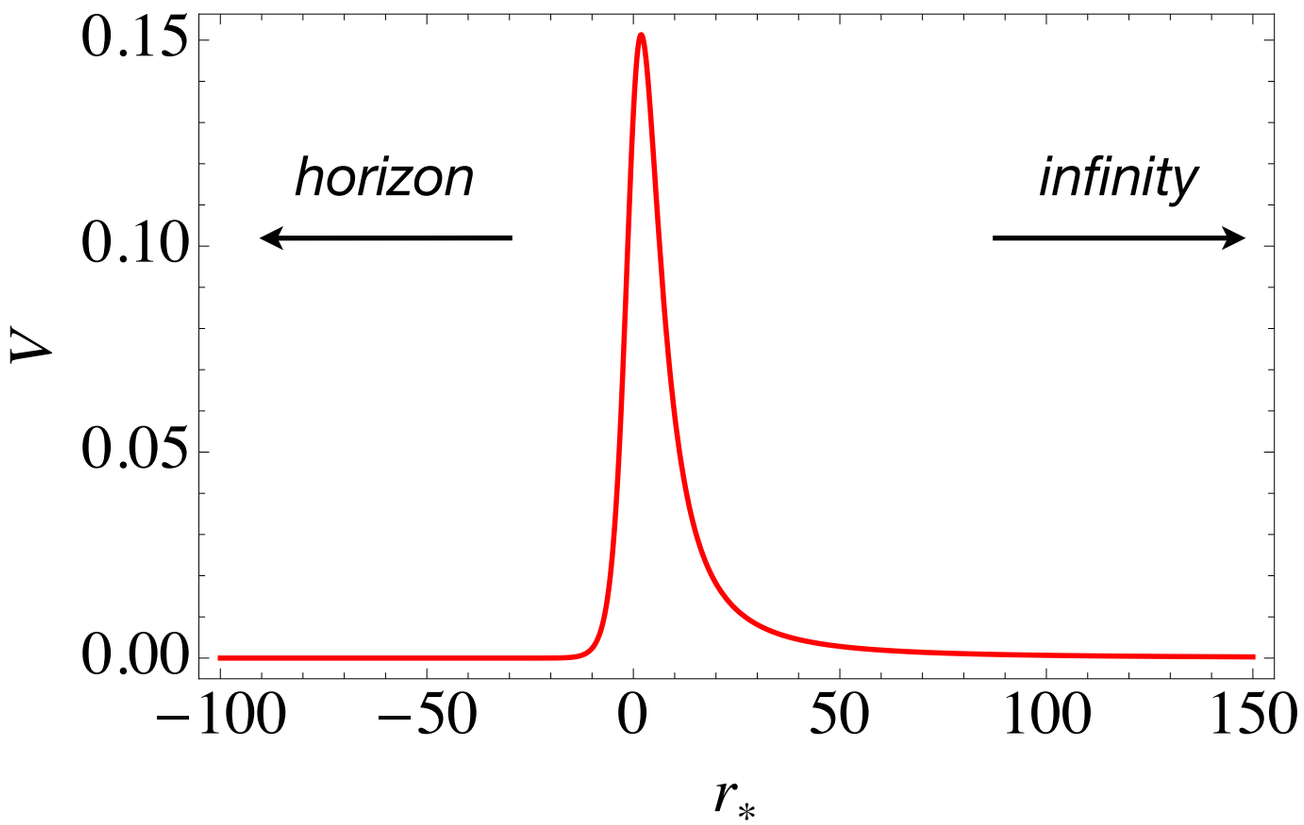


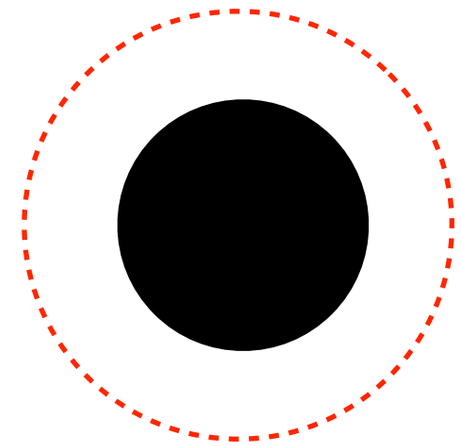


- Physical reasons:
  - additional infinity: perturbations go "down the horizon"
  - potential barrier: gravity effect of black hole

Quasi-Normal Modes of Black Holes

- Homogeneous solution, out-going-wave conditions at both horizon and infinity

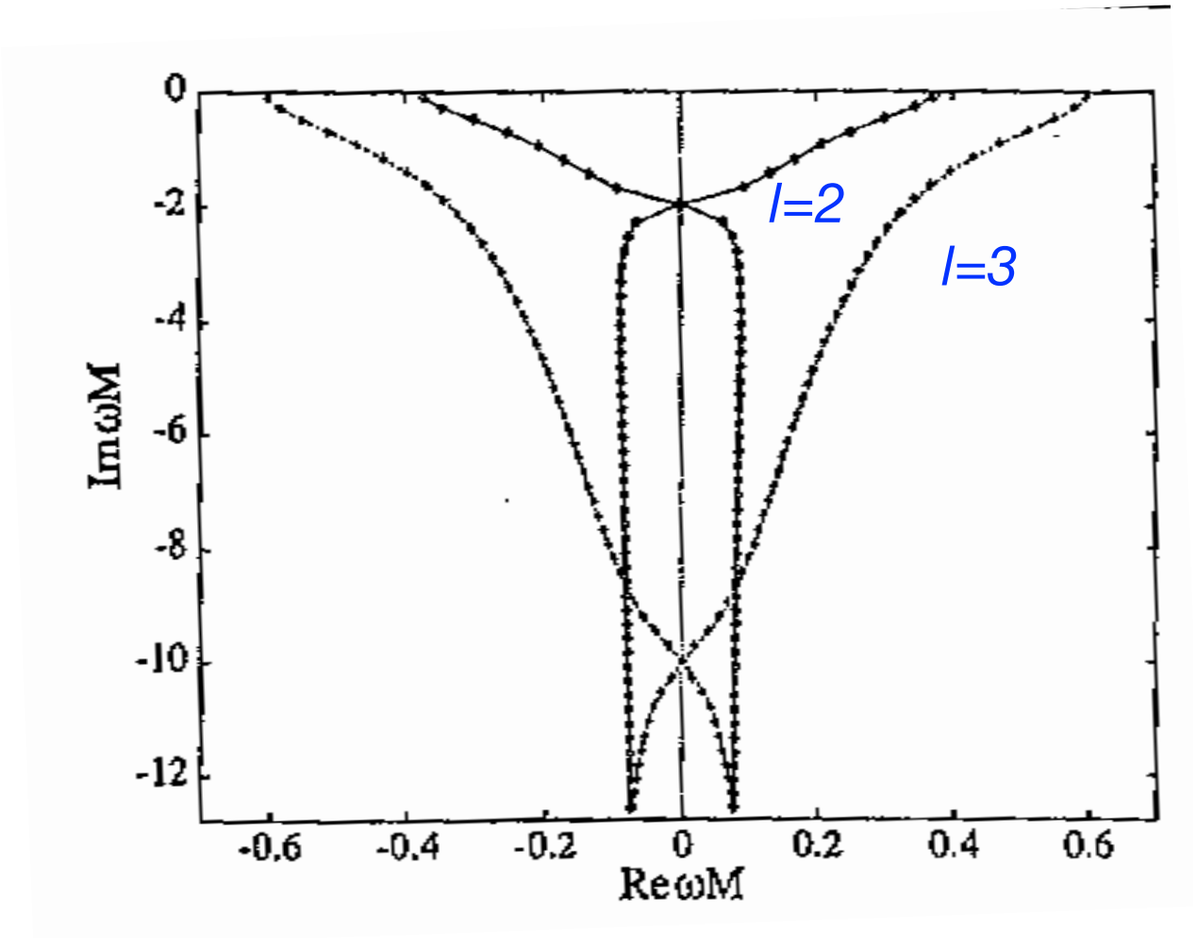




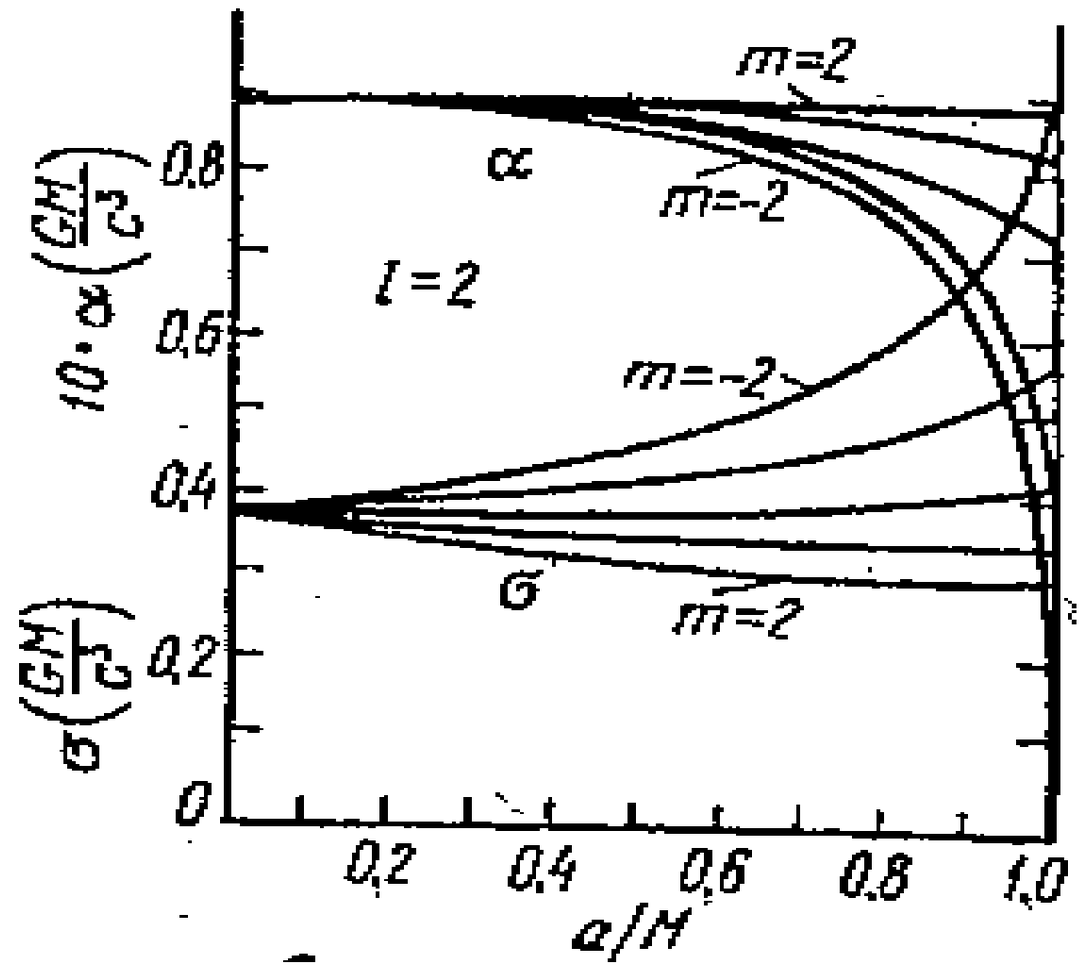
"light ring"

- Typically labeled by *(l,m=-l,...+l,n)*

Quasi-Normal Modes of Black Holes



non-spinning black hole



least damped (2,2) QNM for 0<a/M<1

Lowly damped QNMs are several times higher than ISCO frequency

Theory of Gravitational Collapse

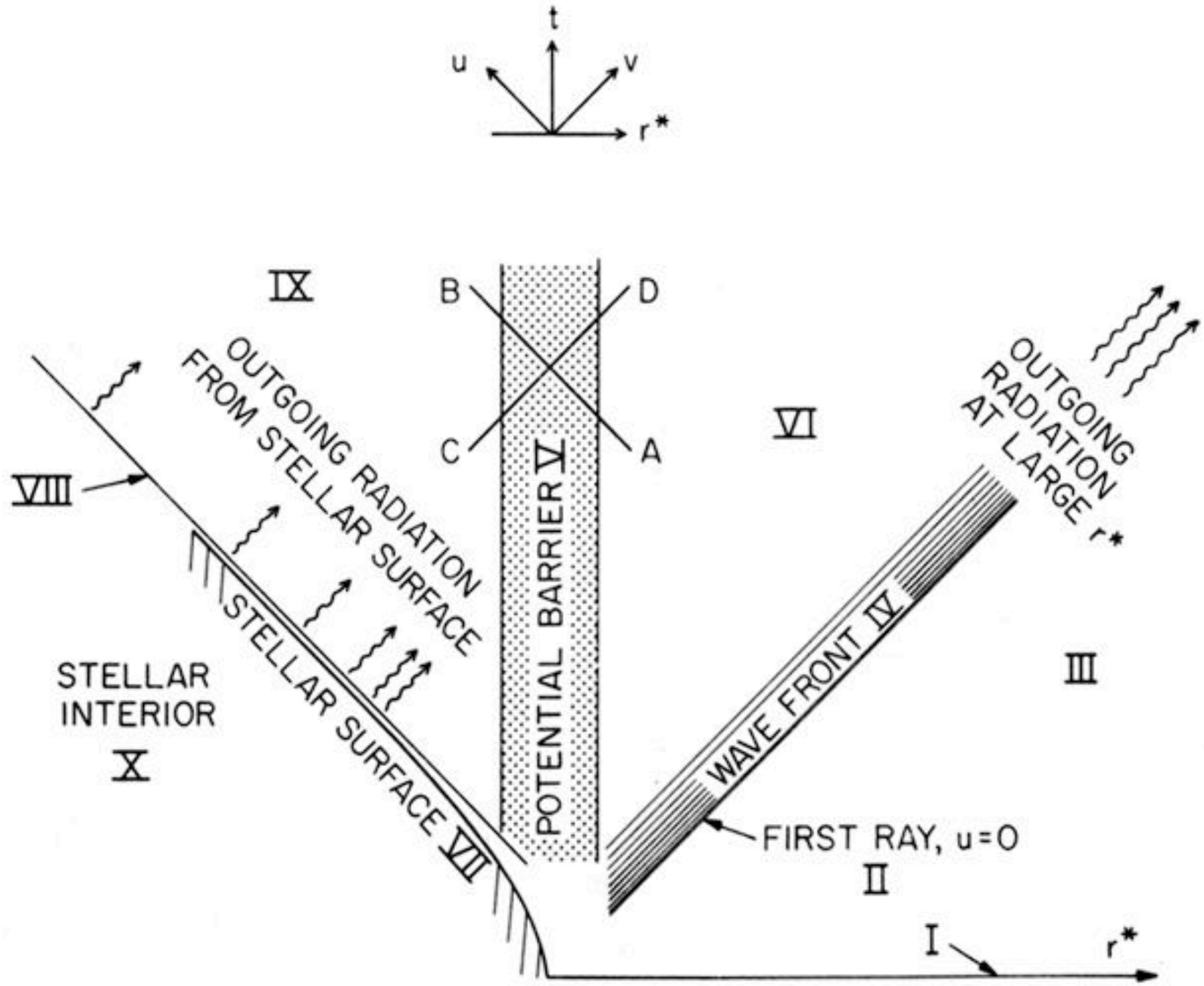
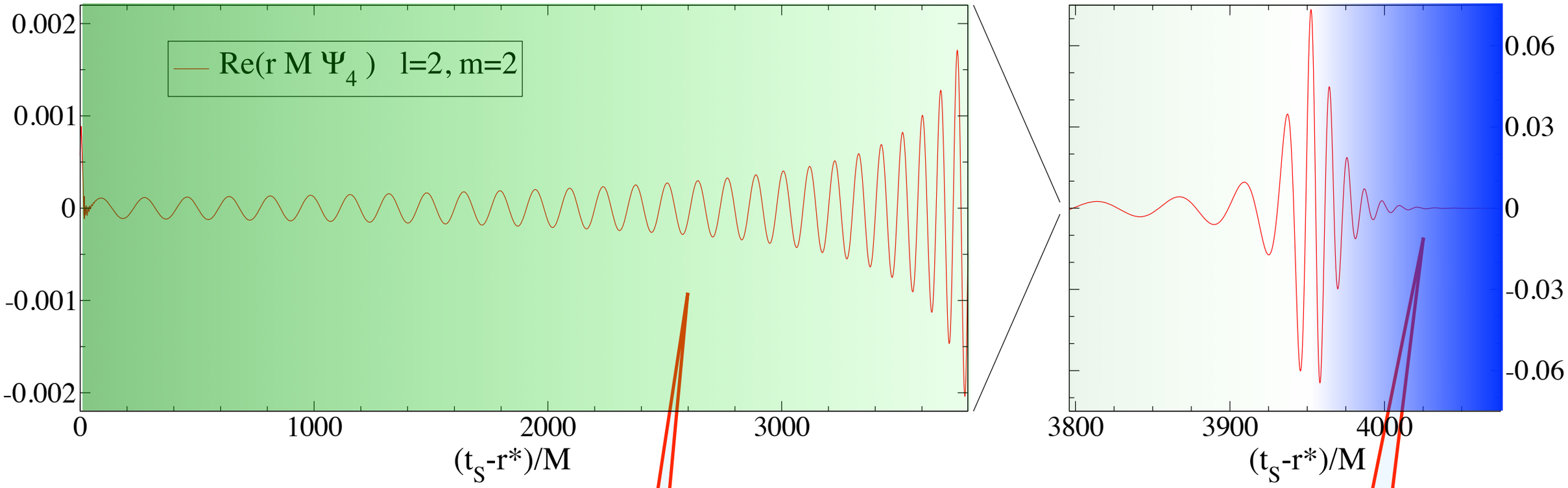


FIG. 4. The "radiation problem" pictured in r, t or u, v coordinates. For explanations and descriptions of features of this diagram, see Table II.

Richard H. Price, *Phys. Rev. D* ***5***, *2419 (1972)*

Full Waveforms of Non-Spinning Binaries



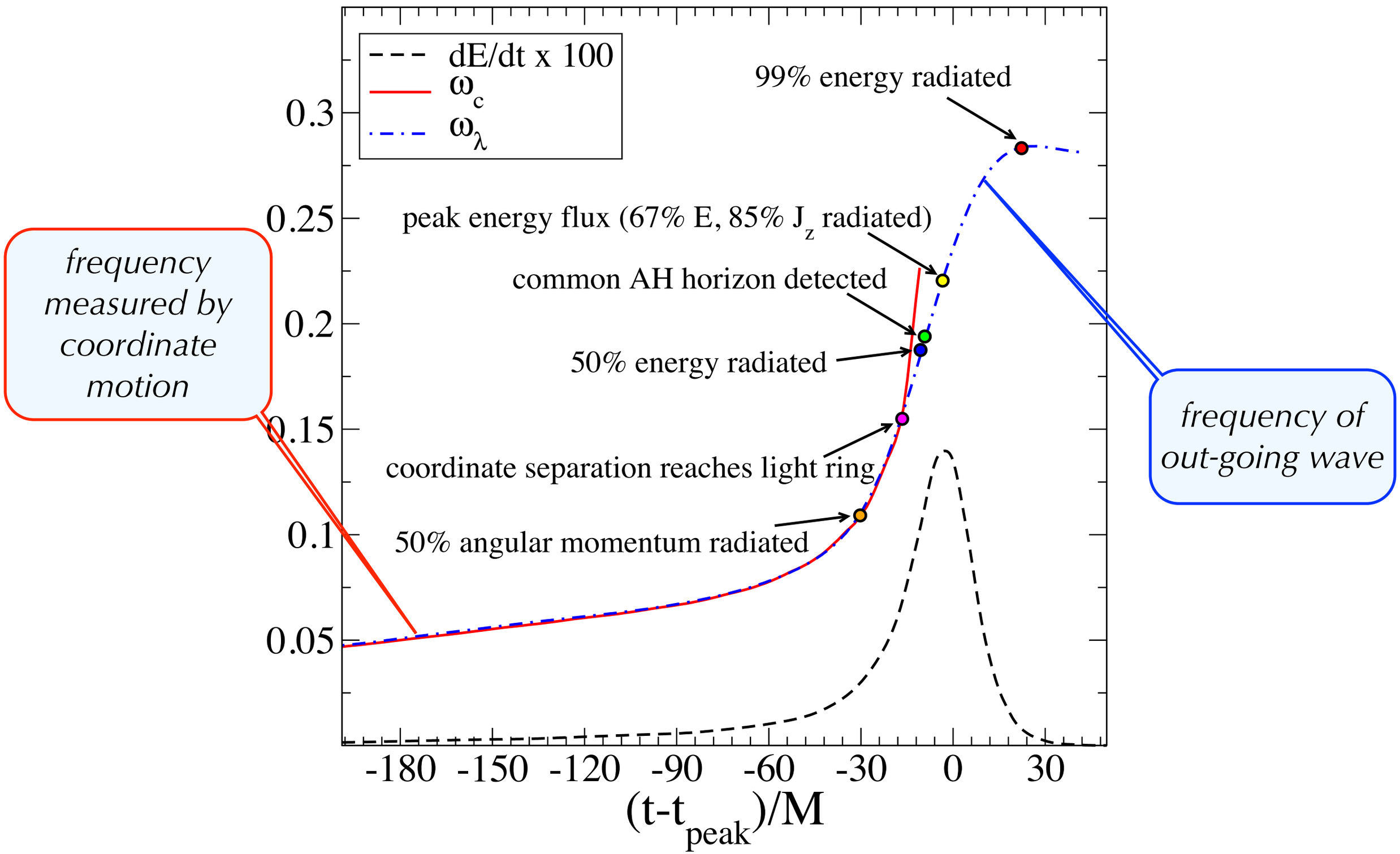
two nearly point particles encircling each other

"Post-Newtonian Expansion"

the final black hole settles down

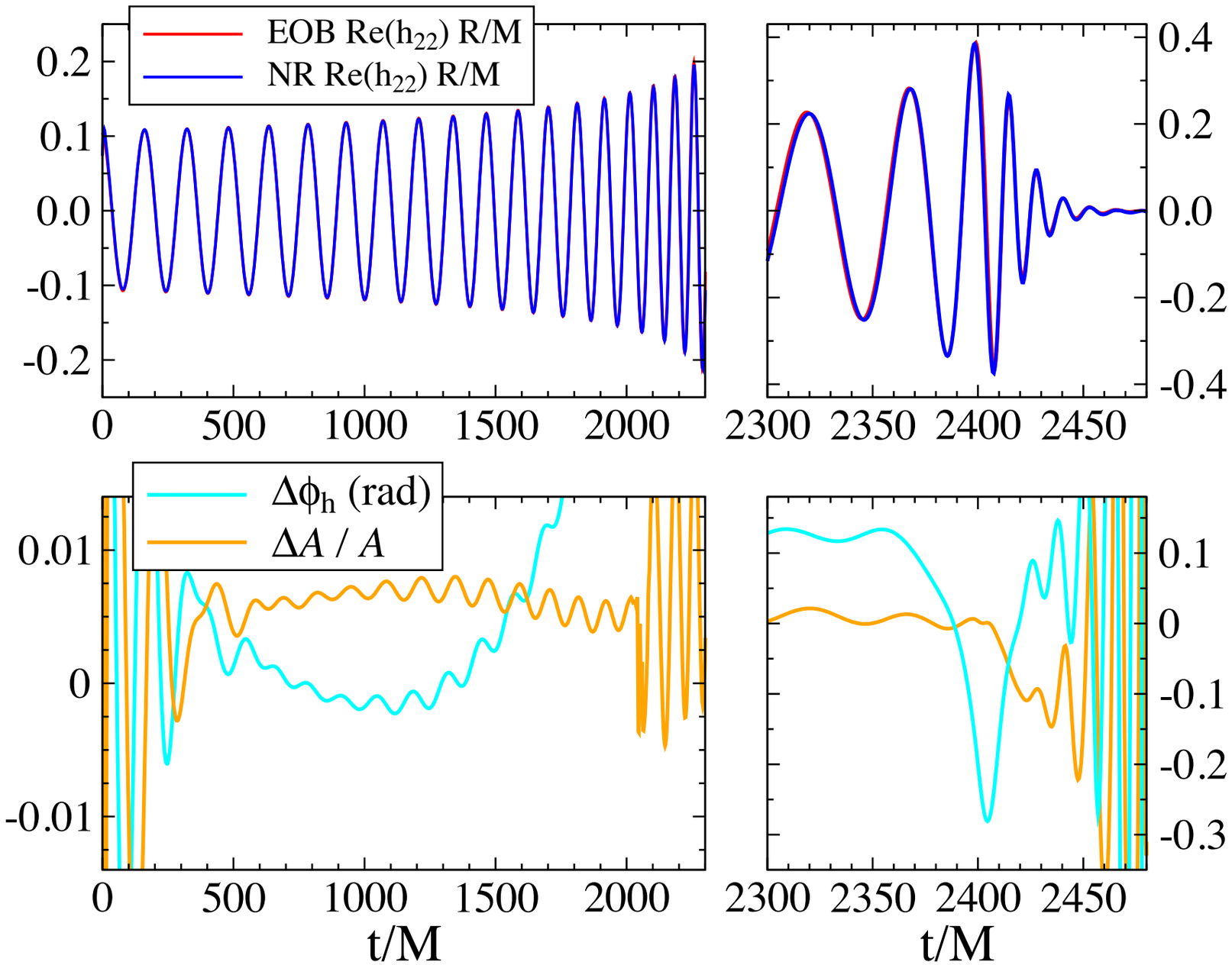
Black-Hole Perturbation Theory

Inspiral, Merger and Ringdown



[Buonanno, Cook & Pretorius, 2007]

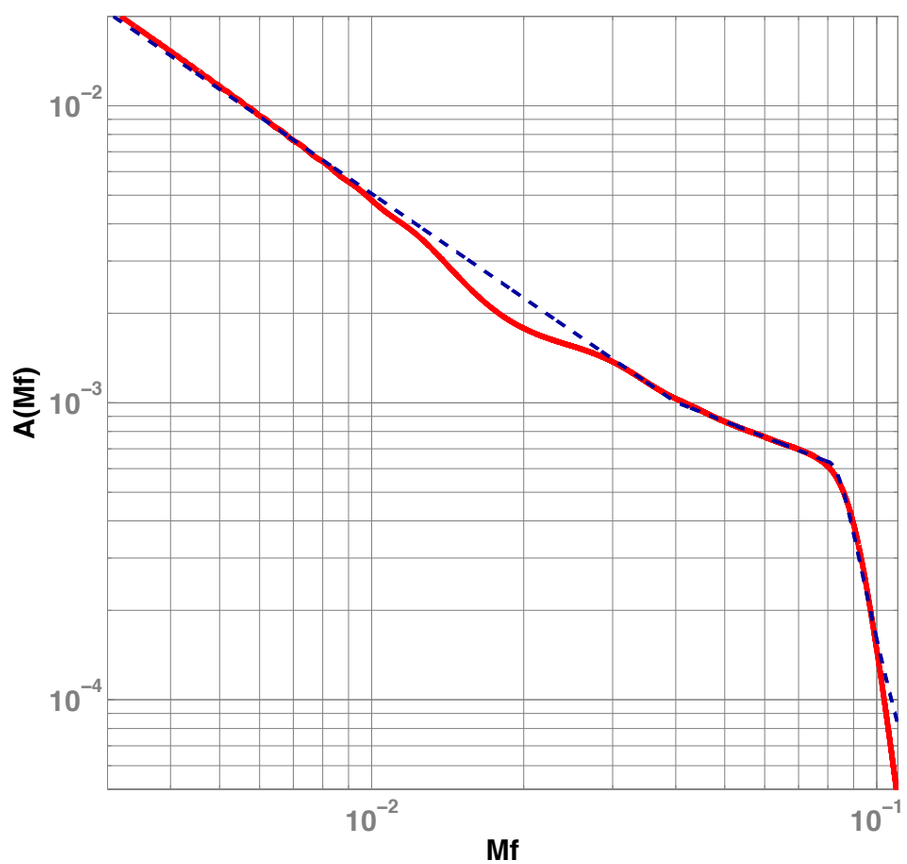
"Attachment" of Quasi-Normal Modes

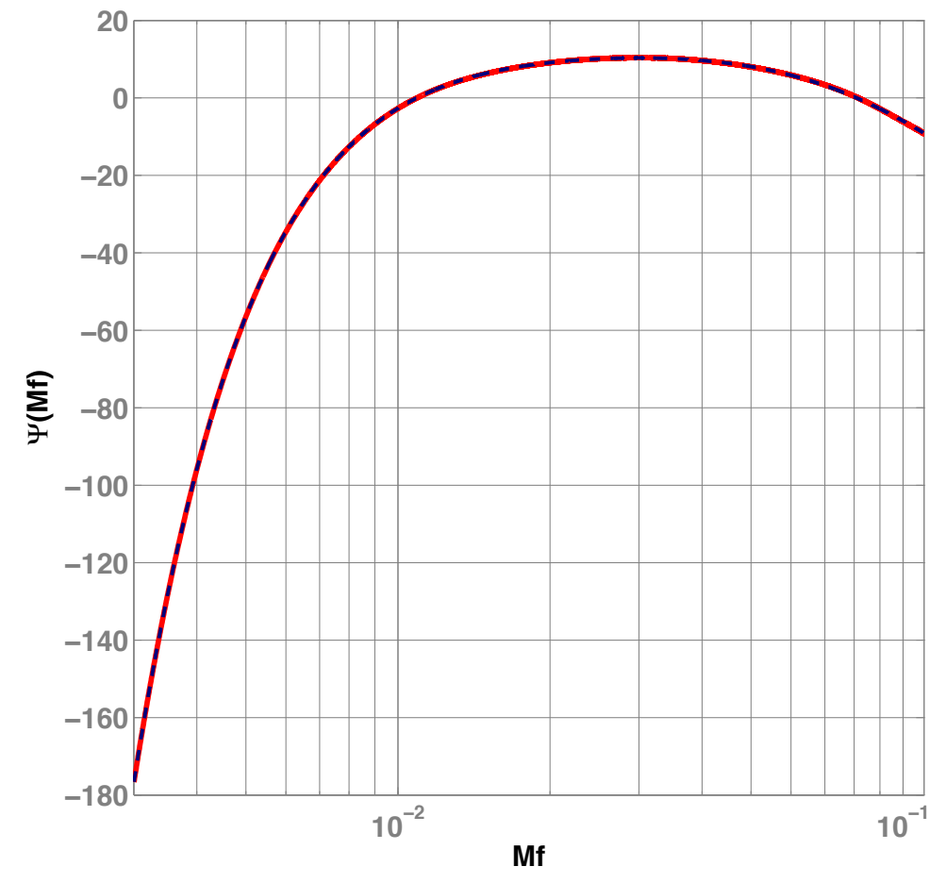


Effective One-Body Waveform, calibrated by numerical simulations with QNMs attached in the time domain

Y. Pan et al., 2009

"Attachment" of Quasi-Normal Modes



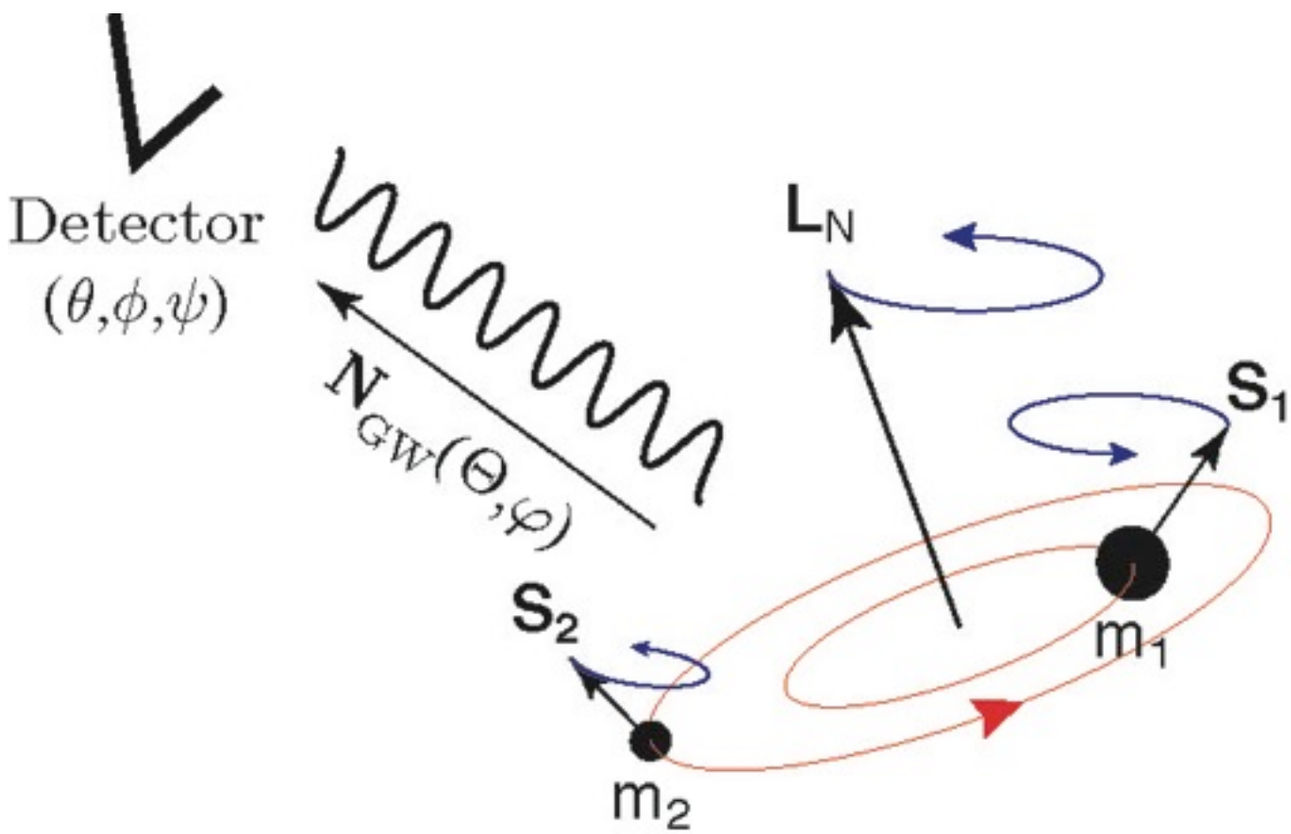


$$A_{\rm eff}(f) \;\equiv\; C \begin{cases} \left(\frac{\pi M f}{a_0\eta^2+b_0\eta+c_0}\right)^{-7/6} & \text{if} \;\; f < \frac{a_0\eta^2+b_0\eta+c_0}{\pi M} \\[2ex] \left(\frac{\pi M f}{a_0\eta^2+b_0\eta+c_0}\right)^{-2/3} & \text{if} \;\; \frac{a_0\eta^2+b_0\eta+c_0}{\pi M} \le f < \frac{a_1\eta^2+b_1\eta+c_1}{\pi M} \\[2ex] w\,\mathcal{L}\left(f,\, \frac{a_1\eta^2+b_1\eta+c_1}{\pi M},\, \frac{a_2\eta^2+b_2\eta+c_2}{\pi M}\right) & \text{if} \;\; \frac{a_1\eta^2+b_1\eta+c_1}{\pi M} \le f < \frac{a_3\eta^2+b_3\eta+c_3}{\pi M}\,, \end{cases}$$

$$\Psi_{\rm eff}(f) \;=\; 2\pi f t_0 + \varphi_0 + \frac{1}{\eta}\sum_{k=0}^{7}(x_k\,\eta^2 + y_k\,\eta + z_k)\,(\pi M f)^{(k-5)/3}\,,$$

P. Ajith et al., 2007: frequency-domain templates

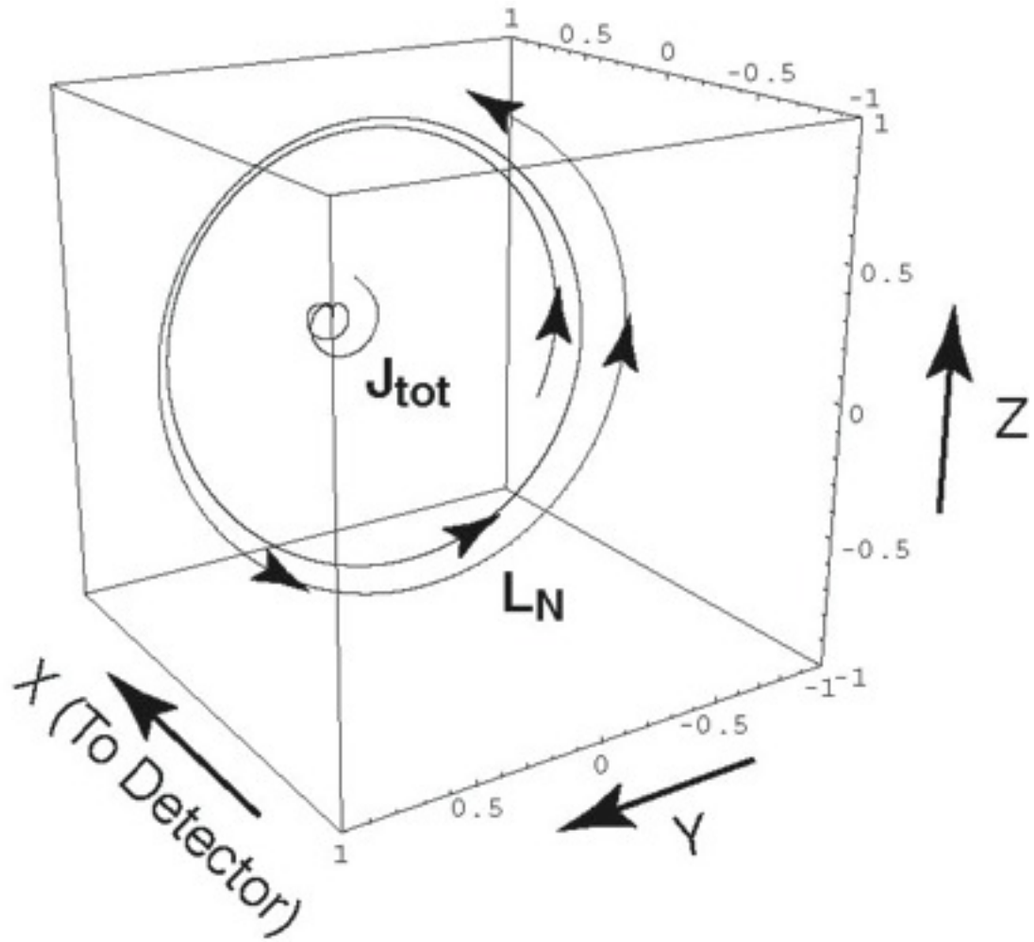
Spin Effects



spin and orbital angular momentum
both precess
under
the influence of each other

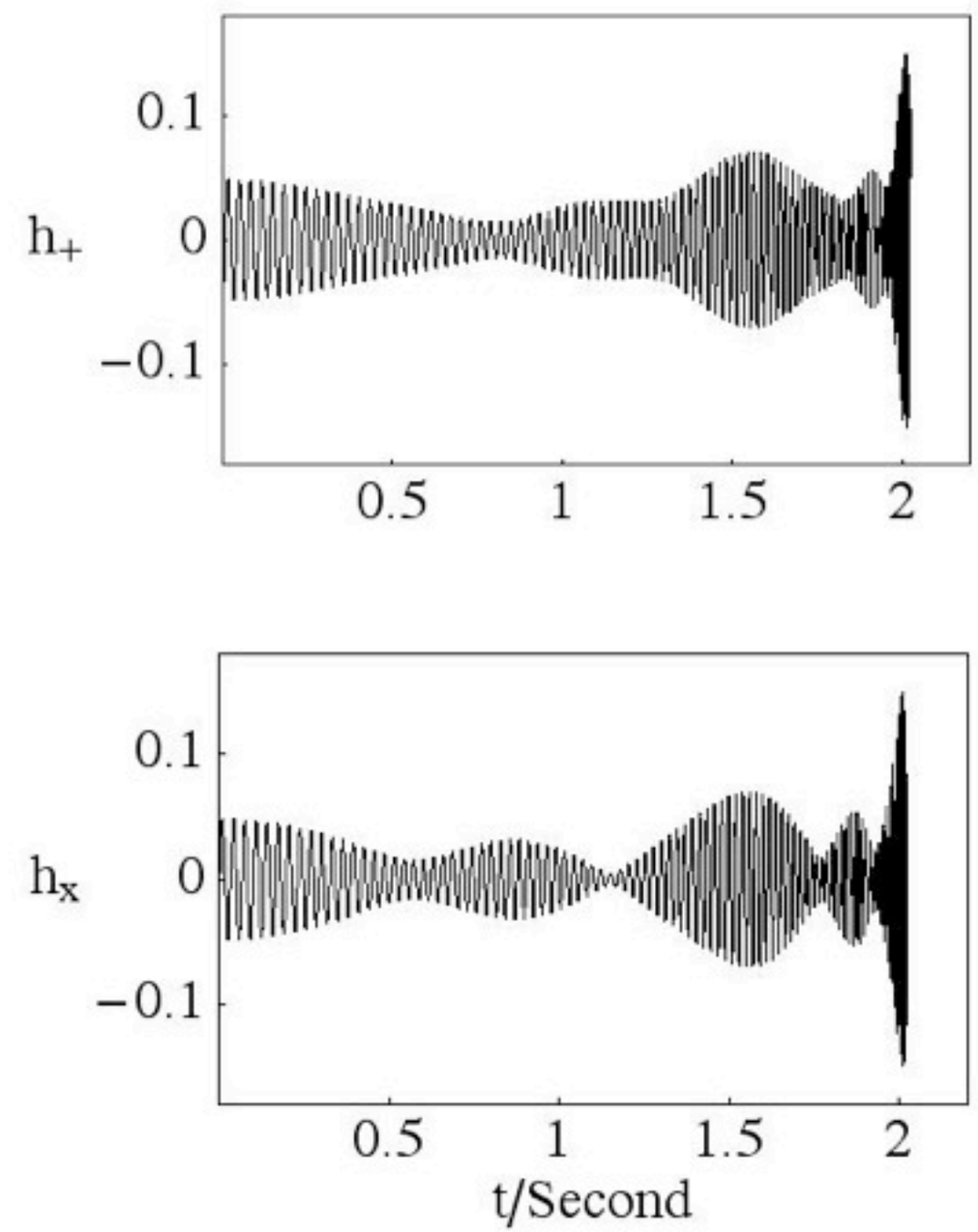
- A **much larger** parameter space:
  - masses (2); $\mathbf{S}_1$ (3), $\mathbf{S}_2$ (3), **L** (2); orbital phase (1), **N** (2); α (1).
  - 14 parameters.
- Reductions
  - total mass (1); overall orientation (3)
  - 10 parameters
- *Additional reductions*
  - if **N** and α are viewed as "one set", then 7 parameters
  - 7 parameters determine all the "*form factors*", 3 determine how they are combined

Effect of Precession



$(20+5)\,M_\odot$, both maximally spining
$\angle(\hat{\mathbf{L}}_{\rm N}, \mathbf{J}_{\rm tot})_0 = 40^\circ$

- Starting at 30 Hz, stopping at 317 Hz
- 46 orbital cycles, 92 GW cycles
- 4.5 modulation cycles



post-Newtonian waveform significantly modulated

Effect of Precession

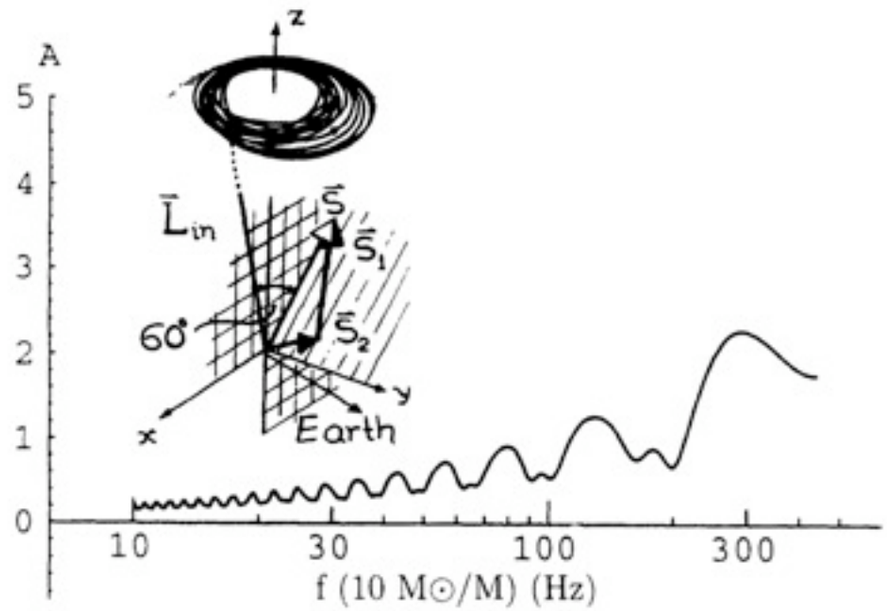


FIG. 14. Same as Fig. 11, but with $M_2/M_1 = 0.3$ (as in Fig. 13) and $S_1 = M_1{}^2$, $S_2 = M_2{}^2$, so the initial $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}$, and $\hat{L}$ are as shown.

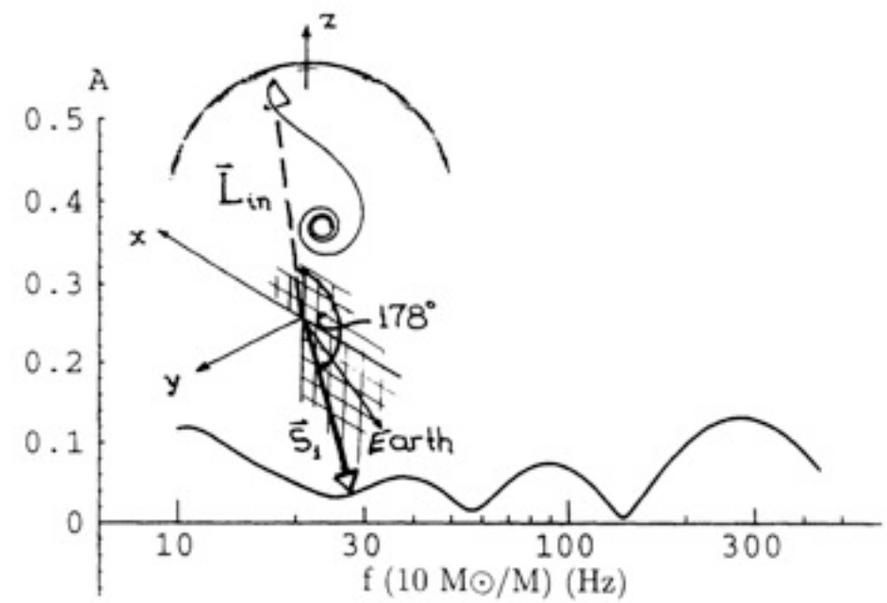


FIG. 17. Same as in Fig. 11, but with $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ nearly antialigned (i.e., separated by an angle of 178° so $\kappa = -0.99939$), and with $M_2/M_1 = 0.13$, $S_1 = M_1{}^2$, and $S_2 = 0$, so the initial $\mathbf{S}_1$ and $\mathbf{L}$ are as shown. The evolution illustrates transitional precession and the subsequent return to simple precession.

Apostolatos, Cutler, Sussman and Thorne, 1994

Hamiltonian with Spin

$$
\begin{aligned}
H^{\mathrm{NNLO}}_{\mathrm{COM\ SO}} = \frac{1}{4r_{12}^5}\Big[&21\sqrt{1-4\eta}(\eta+1)(\mathbf{L}\,\boldsymbol{\Delta}) + \frac{1}{2}(-2\eta^2+33\eta+42)(\mathbf{L}\,\boldsymbol{\Sigma})\Big] \\
&+ \frac{\eta}{32r_{12}^4}\Big[-\sqrt{1-4\eta}\,\big((256+45\eta)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + (314+39\eta)\hat{\mathbf{p}}^2\big)\,(\mathbf{L}\,\boldsymbol{\Delta}) \\
&\quad + \big((-256+275\eta)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + (-206+73\eta)\hat{\mathbf{p}}^2\big)\,(\mathbf{L}\,\boldsymbol{\Sigma})\Big] \\
&+ \frac{\eta}{32r_{12}^3}\Big[\sqrt{1-4\eta}\big(15(\mathbf{n}_{12}\,\hat{\mathbf{p}})^4 + 3(9\eta-4)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 \\
&\quad + 2(22\eta-9)(\hat{\mathbf{p}}^2)^2\big)(\mathbf{L}\,\boldsymbol{\Delta}) - \big(15(2\eta-1)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^4 \\
&\quad + 3(6\eta^2-11\eta+4)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 + 2(5\eta^2-3\eta+2)(\hat{\mathbf{p}}^2)^2\big)(\mathbf{L}\,\boldsymbol{\Sigma})\Big],
\end{aligned}
$$

[Next-Next-Leading Spin-Orbit term,
providing both orbital motion and spin precession
Hartnung et al., 2013]

[Radiation reaction at this order
A. Bohe et al., 2013]

Hamiltonian with Spin

$$
\begin{aligned}
H^{\rm NNLO}_{\rm COM\ SS} = \eta\Bigg\{ & \frac{1}{4r_{12}^5}\Big[(79\eta+105)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) - (63+19\eta)(\hat{\mathbf{S}}_1\,\hat{\mathbf{S}}_2)\Big] \\
& + \frac{1}{r_{12}^4}\Bigg[-\left(\frac{303}{4}\eta(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + \left(\frac{125}{4}\eta+9\right)\hat{\mathbf{p}}^2\right)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) \\
& \quad \left(-\left(18+\frac{25}{4}\eta\right)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + \left(9+\frac{47}{2}\eta\right)\hat{\mathbf{p}}^2\right)(\hat{\mathbf{S}}_1\,\hat{\mathbf{S}}_2) \\
& \quad - \frac{9}{4}(7\eta+4)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2) \\
& \quad + \left(34\eta+\frac{27}{2}\right)(\mathbf{n}_{12}\,\hat{\mathbf{p}})((\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) + (\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2)) \\
& \quad + \frac{3}{2}\sqrt{1-4\eta}(\eta+3)(\mathbf{n}_{12}\,\hat{\mathbf{p}})((\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) - (\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2))\Bigg] \\
& + \frac{1}{r_{12}^3}\Bigg[\frac{1}{8}\Bigg(105\eta^2(\mathbf{n}_{12}\,\hat{\mathbf{p}})^4 + 15\eta(3\eta-2)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 \\
& \quad + \frac{3}{2}(10\eta^2+13\eta-6)(\hat{\mathbf{p}}^2)^2\Bigg)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) \\
& + \frac{1}{8}\Big(-3(8\eta^2-37\eta+12)(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 + (7\eta^2-23\eta+9)(\hat{\mathbf{p}}^2)^2\Big)(\hat{\mathbf{S}}_1\,\hat{\mathbf{S}}_2) \\
& + \frac{1}{4}\Big(9\eta^2(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + \frac{1}{2}(4\eta^2+25\eta-9)\hat{\mathbf{p}}^2\Big)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2) \\
& - \frac{3}{8}\Big(+15\eta^2(\mathbf{n}_{12}\,\hat{\mathbf{p}})^2 + \frac{1}{2}(10\eta^2+21\eta-9)\hat{\mathbf{p}}^2\Big)(\mathbf{n}_{12}\,\hat{\mathbf{p}}) \\
& \quad \times((\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) + (\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2)) \\
& + \frac{9}{16}\sqrt{1-4\eta}(1-2\eta)(\mathbf{n}_{12}\,\hat{\mathbf{p}})((\hat{\mathbf{p}}\,\hat{\mathbf{S}}_1)(\mathbf{n}_{12}\,\hat{\mathbf{S}}_2) - (\mathbf{n}_{12}\,\hat{\mathbf{S}}_1)(\hat{\mathbf{p}}\,\hat{\mathbf{S}}_2))\Bigg]\Bigg\}.
\end{aligned}
$$

[Radiation reaction at this order not yet developed?]

[Next-Next-Leading Spin-Spin term, Hartnung et al., 2013]

Comparison with Numerical Relativity

A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy

Abdul H. Mroué,[1] Mark A. Scheel,[2] Béla Szilágyi,[2] Harald P. Pfeiffer,[1,3] Michael Boyle,[4] Daniel A. Hemberger,[4] Lawrence E. Kidder,[4] Geoffrey Lovelace,[5,2] Serguei Ossokine,[1,6] Nicholas W. Taylor,[2] Anıl Zenginoğlu,[2] Luisa T. Buchman,[2] Tony Chu,[1] Evan Foley,[5] Matthew Giesler,[5] Robert Owen,[7] and Saul A. Teukolsky[4]

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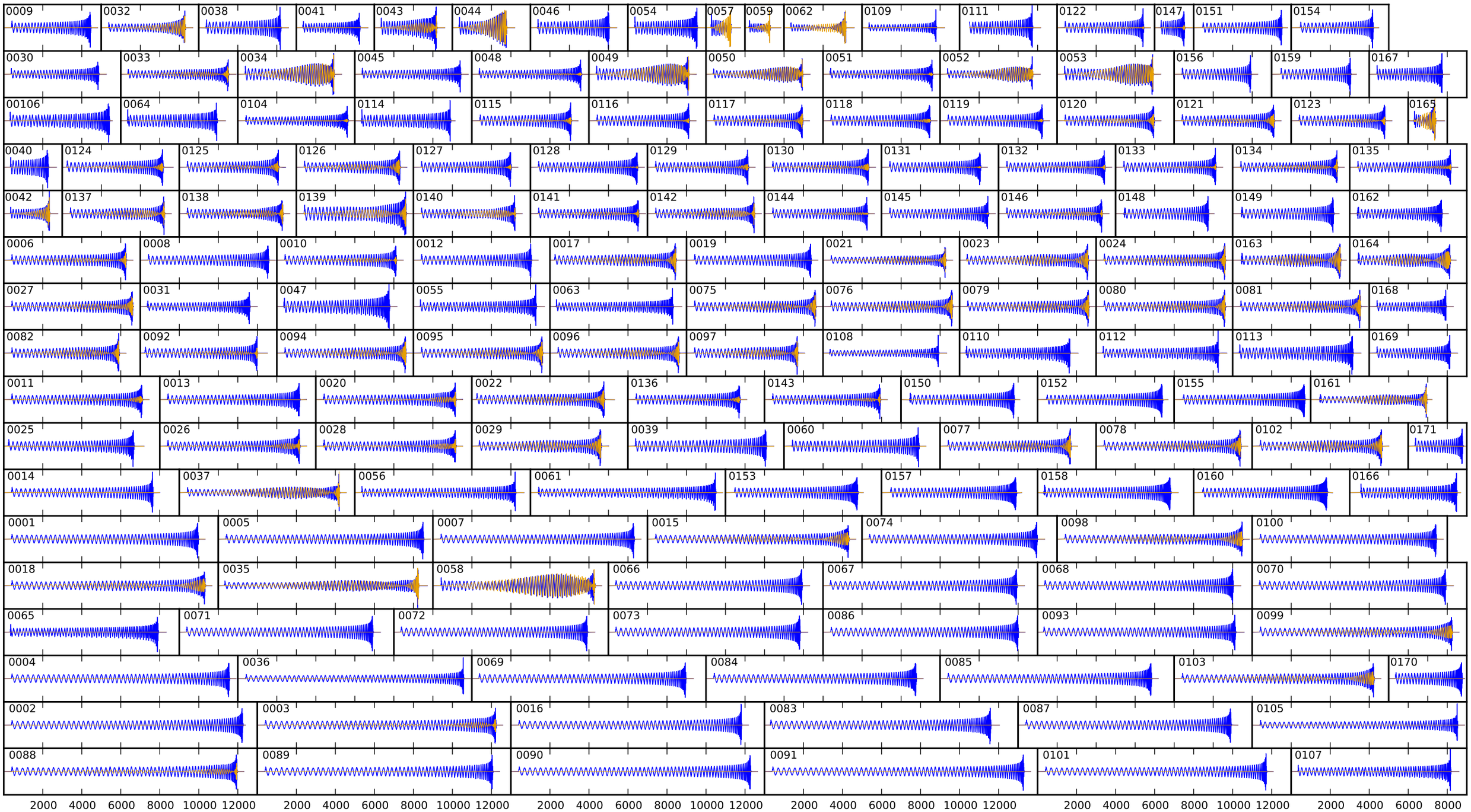
(Dated: June 4, 2013)

Comparison with Numerical Relativity

A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy



Beyond Spins

- Tidal interaction between BHs can be viewed as GWs propagating down the horizon
  [Fang & Lovelace 2005; Li & Lovelace, 2008; earlier work by Price & Pullin, K. Alvi]

deforming each other ...

GW down the horizon

causing Masses & Spins of BHs to evolve slightly

Summary

- Semi-analytical modeling can provide approximate waveforms fast

- ... also providing physical interpretations to various components of the waveform
  - yet higher-order corrections do not have simple interpretations

- For non-spinning binaries (at least for low multipole waves), it is straightforward to *calibrate* existing approximations by NR computations.
  - complexity is low because only one parameter, the mass ratio.

- For spinning binaries
  - complexity is much higher due to high # of parameters
  - comparison with NR is on-going.