Reduced order modelling in multi-parametrized system: applications to inverse problems and optimal control

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Outline

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   - Stokes constraint: a numerical test (L2) and a data assimilation problem for blood flows (L3)

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   - Brezzi-Rappaz-Raviart theory to obtain error bounds
   - Benchmark test: vorticity minimization (NL1)
Complex parametrized systems

\[ \frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla \pi = f \quad \text{in } \Omega(\mu) \]
\[ \text{div } v = 0 \quad \text{in } \Omega(\mu) \]
\[ \frac{\partial C}{\partial t} - \nabla \cdot (\mu_c \nabla C) + v \cdot \nabla C = 0 \quad \text{in } \Omega(\mu) \]

- **Parametrized Simulation Problems**

\[ \min J(y, u; \mu) = F(y; \mu) + G(u; \mu) \]
subject to
\[ -\nu(\mu) \Delta y + b(\mu) \cdot \nabla y = 0 \quad \text{in } \Omega(\mu) \]
\[ y = u \quad \text{on } \Gamma(\mu) \]

- **Parametrized Data assimilation and Inverse Problems**

\[ \min J(v, \pi, u; \mu) = F(v, \pi; \mu) + G(u; \mu) \]
subject to
\[ -\nu \Delta v + (v \cdot \nabla) v + \nabla \pi = 0 \quad \text{in } \Omega(\mu) \]
\[ \text{div } v = 0 \quad \text{in } \Omega(\mu) \]
\[ v = u \quad \text{on } \Gamma_D(\mu) \]

- **Parametrized Optimization Problems**
The main obstacle to make mathematical models extensively useful and reliable in the clinical context is that they have to be personalized.

Many quantities required by the numerical simulations cannot be always obtained through direct measurements and thus need to be estimated using the available clinical measurements.

The ultimate goal would be to optimize the therapeutic intervention depending on the patient attributes.
Complexity in haemodynamics

- **Parametrized Simulation Problems**

- Many quantities required by the numerical simulations cannot be always obtained through direct measurements and thus need to be estimated using the available clinical measurements.

- The ultimate goal would be to optimize the therapeutic intervention depending on the patient attributes.
Complexity in haemodynamics

- **Parametrized Simulation Problems**

- **Parametrized Data assimilation and Inverse Problems**

- The ultimate goal would be to optimize the therapeutic intervention depending on the patient attributes
Complexity in haemodynamics

- Parametrized Simulation Problems
- Parametrized Data assimilation and Inverse Problems
- Parametrized Optimization Problems
Models and problems

- surface reconstruction of blood flow profiles
- inverse problems: reconstruction of boundary conditions by experimental measures/observations
- flow control: vorticity reduction by suction/injection of fluid through the boundary

**Steady state system**: advection-diffusion, Stokes or Navier-Stokes equations

**Control variables**: distributed in the domain or along the boundary

**Parameters**: they can be physical/geometrical quantities describing the state system or related to observation measurements in the cost functional
Optimal control problems [Lions, 1971]

In general, an optimal control problem (OCP) consists of:

- a control function $u$, which can be seen as an input for the system,
- a controlled system, i.e. an input-output process: $\mathcal{E}(y, u) = 0$, being $y$ the state variable
- an objective functional to be minimized: $\mathcal{J}(y, u)$
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- an objective functional to be minimized: $\mathcal{J}(y, u)$

\[ \begin{array}{ccc}
  \text{STATE PROBLEM} & \rightarrow & \text{Output} \\
  u & \rightarrow & y(u) \\
  \downarrow & \text{Optimization:} & \downarrow \\
  & \text{update control } u & \\
\end{array} \]

\[ \mathcal{J}(y, u) \]

\[ \text{find the optimal control } u^* \text{ and the state } y(u^*) \text{ such that the cost functional } \mathcal{J}(y, u) \text{ is minimized subject to } \mathcal{E}(y, u) = 0 \]
Optimal control problems [Lions, 1971]

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- an objective functional to be minimized: $\mathcal{J}(y, u)$

\begin{equation}
\text{find the optimal control } u^* \text{ and the state } y(u^*) \text{ such that the cost functional } \mathcal{J}(y, u) \text{ is minimized subject to } \mathcal{E}(y, u) = 0
\end{equation}

We restrict attention to:

- quadratic cost functionals, e.g. $\mathcal{J}(y, u) = \frac{1}{2} \| y - y_d \|^2 + \frac{\alpha}{2} \| u \|^2$
**Parametrized optimal control problems**

A parametrized optimal control problem (OCP\(_\mu\)) consists of:

- a control function \(u(\mu)\), which can be seen as an input for the system,
- a controlled system, i.e. an input-output process:
  \[
  E(y(\mu), u(\mu); \mu) = 0,
  \]
- an objective functional to be minimized:
  \[
  J(y(\mu), u(\mu); \mu)
  \]

\[
\text{given } \mu \in \mathcal{D}, \text{ find the optimal control } u^*(\mu) \text{ and the state } y^*(\mu) \text{ such that the cost functional } J(y(\mu), u(\mu); \mu) \text{ is minimized subject to } E(y(\mu), u(\mu); \mu) = 0
\]

where \(\mu \in \mathcal{D} \subset \mathbb{R}^p\) denotes a \(p\)-vector whose components can represent:

- coefficients in boundary conditions
- geometrical configurations
- physical parametrization
- data (observation)
Reduction strategies for **Parametrized** Optimal Control Problems

**PROBLEM**: given \( \mu \in D \subset \mathbb{R}^p \),

\[
\min_{y,u} J(y, u; \mu)
\]

\text{s.t.} \quad \mathcal{E}(y, u; \mu) = 0

The computational effort may be unacceptably high and, often, unaffordable when

- performing the optimization process for many different parameter values
  
  (\textit{many-query} context)

- for a given new configuration, we want to compute the solution in a rapid way
  
  (\textit{real-time} context)

**Goal**: to achieve the **accuracy** and **reliability** of a high fidelity approximation

but at greatly **reduced cost** of a **low order model**
Parametrized control problems vs Parametric Control

\[
\min J(y, u; \mu) = F(y; \mu) + G(u; \mu)
\]
subject to
\[
-\nu(\mu)\Delta y + b(\mu) \cdot \nabla y = u \quad \text{in } \Omega(\mu)
\]
\[
y = g \quad \text{on } \Gamma_D(\mu)
\]
\[
-\nu(\mu)\nabla y \cdot n = 0 \quad \text{on } \Gamma_N(\mu)
\]

Control variables: distributed in the domain or along the boundary

\[
\min J(v, \pi, u; \mu) = F(v, \pi; \mu) + G(u; \mu)
\]
subject to
\[
-\nu\Delta v + (v \cdot \nabla)v + \nabla \pi = 0 \quad \text{in } \Omega(\mu)
\]
\[
div v = 0 \quad \text{in } \Omega(\mu)
\]
\[
v = u \quad \text{on } \Gamma_D(\mu)
\]

In many previous works the idea was to reduce the complexity of the problem by introducing a Parametric Control, e.g. \( u(\mu_c) = \sum \mu_i^c h_i(x) \)

- We need to introduce further parameters \([\mu, \mu^c]\)
- How many terms to obtain good approximation of the control space?
- Certification of the optimum still missing (need reduced model of the adjoint)
- A priori constraints on the range of variations of the parameters \(\mu_c\)
Previous works and improvements on parametrized optimal control

- early works by Ito and Ravindran [1998 and 2001] using either (a preliminary version of) RB or POD methods

- the RB method has been applied to parametrized linear-quadratic advection diffusion optimal control problems in different contexts: [Quarteroni et al., 2007], [Tonn et al., 2011], [Grepl & Karcher, 2011] considering low-dimensional control variable.

- we aim at developing a reduced framework that enables to handle with general control functions, i.e. infinite dimensional distributed and/or boundary control functions.

- The need of efficient and rigorous a posteriori error estimates – necessary both for constructing the reduced order model and for assessing its accuracy – is still partially unresolved. Previous preliminary works by [Tonn et al., 2011] [Dede’, 2010] [Grepl & Karcher, 2011, 2012]
Main ingredients: **linear** state equation case

We build the Reduced Basis (RB) approximation directly on the optimality (KKT) system:

- we firstly recast the problem in the framework of **saddle-point** problem [Gunzburger & Bochev, 2004]

- we then apply the well-known **Brezzi-Babuška theory** [Brezzi & Fortin, 1991]

This way we can exploit the analogies with the already developed theory of RB method for **Stokes-type problems** [Rozza & Veroy, 2007] [Rozza et al., n.d.] [Gerner & Veroy, 2012]

The usual ingredients of the RB methodology are provided:

- Galerkin projection onto a **low-dimensional space** of basis functions properly selected by a greedy algorithm for optimal parameters sampling;

- affine parametric dependence $\rightarrow$ **Offline-Online** computational procedure [EIM];

- an efficient and rigorous **a posteriori** error estimation on the solution variables as well as on the cost functional.

\[
M^N = \{ U^N(\mu) \in X^N : \mu \in D \} \\
X^N = \text{span}\{ U^N(\mu^i) , i = 1, \ldots, N \} 
\]
Main ingredients: **nonlinear** state equation (Navier-Stokes) case

Again, we work directly on the optimality system, in this case a **nonlinear** system of PDEs

- **Newton-SQP** method: sequence of saddle-point problems featuring the same structure of the optimality system in the linear case [Ito & Kunisch, 2008]

- we then apply the **Brezzi-Rappaz-Raviart theory** [Brezzi, Rappaz, Raviart, 1980]

This way we can exploit the analogies with the already developed theory of RB method for nonlinear equations (in particular Navier-Stokes) [Patera, Veroy, R., Deparis, Manzoni]

The usual ingredients of the RB methodology are provided:

- **Galerkin projection** onto a low-dimensional space of basis functions properly selected by a greedy algorithm for optimal parameters sampling;

- affine parametric dependence → **Offline-Online** computational procedure [EIM];

- an efficient and rigorous a posteriori error estimation on the solution variables as well as on the cost functional [in progress].

\[
\mathcal{M}^N = \{u^N(\mu) \in X^N : \mu \in \mathcal{D}\}
\]

\[
X^N = \text{span}\{u^N(\mu^i), i = 1, \ldots, N\}
\]
Optimality system

Let \( x = (y, u) \) be the optimization variable (state and control variables),

\[
\begin{align*}
\text{given } \mu \in \mathcal{D} \subset \mathbb{R}^p, \quad \min_{x \in X} \mathcal{J}(x; \mu) \quad \text{s.t.} \quad \mathcal{E}(x; \mu) = 0 \text{ in } Q'
\end{align*}
\]

Lagrangian functional:

\[
\mathcal{L}(x, p; \mu) = \mathcal{J}(x, \mu) + \langle \mathcal{E}(x, \mu), p \rangle,
\]

By requiring the first derivatives to vanish we obtain the optimality (KKT) system

\[
\begin{align*}
\mathcal{J}_x(x; \mu) + \mathcal{E}_x(x; \mu)^* p &= 0 \\
\mathcal{E}(x; \mu) &= 0
\end{align*}
\]
Optimality system

Let \( x = (y, u) \) be the optimization variable (state and control variables),

\[
\text{given } \mu \in D \subset \mathbb{R}^p, \quad \min_{x \in X} J(x; \mu) \quad \text{s.t.} \quad E(x; \mu) = 0 \text{ in } Q'
\]

Lagrangian functional:

\[
L(x, p; \mu) = J(x, \mu) + \langle E(x, \mu), p \rangle,
\]

By requiring the first derivatives to vanish we obtain the optimality (KKT) system

\[
\begin{align*}
J_x(x; \mu) + E_x(x; \mu)^* p &= 0 \\
E(x; \mu) &= 0
\end{align*}
\]

**Linear state equation:** \( E(\cdot; \mu): X \to Q' \) is linear,

\[
\begin{align*}
J(x; \mu) &= \frac{1}{2} \langle A(\mu)x, x \rangle - \langle f(\mu), x \rangle \\
E(x; \mu) &= B(\mu)x - g(\mu) \quad \implies \quad E_x(x; \mu)^* = B^*(\mu) \text{ independent of } x
\end{align*}
\]

Algebraic formulation:

\[
\begin{pmatrix}
A(\mu) & B^T(\mu) \\
B(\mu) & 0
\end{pmatrix}
\begin{pmatrix}
x(\mu) \\
p(\mu)
\end{pmatrix}
= \begin{pmatrix}
F(\mu) \\
G(\mu)
\end{pmatrix}
\]
Optimality system

Let \( x = (y, u) \) be the optimization variable (state and control variables),

\[
\text{given } \mu \in \mathcal{D} \subset \mathbb{R}^p, \quad \min_{x \in X} J(x; \mu) \quad \text{s.t.} \quad \mathcal{E}(x; \mu) = 0 \text{ in } Q'
\]

Lagrangian functional:
\[
\mathcal{L}(x, p; \mu) = J(x, \mu) + \langle \mathcal{E}(x, \mu), p \rangle,
\]

By requiring the first derivatives to vanish we obtain the optimality (KKT) system

\[
\begin{align*}
\mathcal{J}_x(x; \mu) + \mathcal{E}_x(x; \mu)^* p &= 0 \\
\mathcal{E}(x; \mu) &= 0
\end{align*}
\]

Nonlinear state equation: \( \mathcal{E}(:,:,\mu) : X \rightarrow Q' \) is nonlinear. Newton's method on the optimality system: for \( k = 1, 2, \ldots \)

solve for \( (s_x^k, s_p^k) \)
\[
\begin{align*}
\mathcal{L}_{xx}(x^k, p^k; \mu) s_x^k + \mathcal{E}_x(x^k; \mu)^* s_p^k &= -\mathcal{L}_x(x^k, p^k; \mu) \\
\mathcal{E}_x(x^k; \mu) s_x^k &= -\mathcal{E}(x^k, \mu)
\end{align*}
\]

update
\[
x^{k+1} = x^k + s_x^k, \quad p^{k+1} = p^k + s_p^k
\]
Optimality system

Let \( x = (y, u) \) be the optimization variable (state and control variables),

\[
\begin{aligned}
\text{given } \mu \in \mathcal{D} \subset \mathbb{R}^p, \quad \min_{x \in X} \mathcal{J}(x; \mu) \quad \text{s.t.} \quad \mathcal{E}(x; \mu) = 0 \quad \text{in } Q',
\end{aligned}
\]

Lagrangian functional:

\[
\mathcal{L}(x, p; \mu) = \mathcal{J}(x, \mu) + \langle \mathcal{E}(x, \mu), p \rangle,
\]

By requiring the first derivatives to vanish we obtain the optimality (KKT) system

\[
\begin{aligned}
\mathcal{J}_x(x; \mu) + \mathcal{E}_x(x; \mu)^* p &= 0 \\
\mathcal{E}(x; \mu) &= 0
\end{aligned}
\]

**Nonlinear state equation:** \( \mathcal{E}(\cdot; \mu): X \to Q' \) is nonlinear. Newton’s method on the optimality system: for \( k = 1, 2, \ldots \)

solve for \((s^k_x, s^k_p)\)

\[
\begin{pmatrix}
A^k(\mu) & B^k(\mu)^T \\
B^k(\mu) & 0
\end{pmatrix}
\begin{pmatrix}
s^k_x(\mu) \\
s^k_p(\mu)
\end{pmatrix} = \begin{pmatrix}
F^k(\mu) \\
G^k(\mu)
\end{pmatrix}
\]

update

\[
x^{k+1} = x^k + s^k_x, \quad p^{k+1} = p^k + s^k_p
\]
The abstract optimization problem

**Notation:**

\( y, z \in Y \) \text{ state space} \quad u, v \in U \text{ control space} \\
\( p, q \in Q \) \text{ (\( \equiv Y \)) adjoint space} \quad \mathcal{Z} \text{ observation space s.t. } Y \subset \mathcal{Z} \\

**Parametrized optimal control problem:** given \( \mu \in \mathcal{D} \)

\[
\text{minimize} \quad J(y, u; \mu) = \frac{1}{2} m(y - y_d(\mu), y - y_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu) \\
\text{s.t.} \quad a(y, q; \mu) = c(u, q; \mu) + \langle G(\mu), q \rangle \quad \forall q \in Q.
\]
The abstract optimization problem: saddle-point formulation

**Notation:**
- $y, z \in Y$: state space
- $u, v \in U$: control space
- $p, q \in Q \equiv Y$: adjoint space
- $\mathcal{Z}$: observation space s.t. $Y \subset \mathcal{Z}$

**Parametrized optimal control problem:** given $\mu \in \mathcal{D}$

$$
\begin{align*}
\text{minimize} & \quad J(y, u; \mu) = \frac{1}{2} m(y - y_d(\mu), y - y_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu) \\
\text{s.t.} & \quad a(y, q; \mu) = c(u, q; \mu) + \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{align*}
$$

Let $X \equiv Y \times U$ be the state and control space, the constrained optimization problem can be recast in the form:

**Saddle-point formulation:** given $\mu \in \mathcal{D}$

$$
\begin{align*}
\left\{ \begin{array}{l}
\min \mathcal{J}(x; \mu) = \frac{1}{2} A(x, x; \mu) - \langle F(\mu), x \rangle, \\
B(x, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{array} \right.
\end{align*}
$$

where

$$
\begin{align*}
A(x, w; \mu) &= m(y, z; \mu) + \alpha n(u, v; \mu), \\
\langle F(\mu), w \rangle &= m(y_d(\mu), z; \mu) \\
B(w, q; \mu) &= a(z, q; \mu) - c(v, q; \mu)
\end{align*}
$$

**notation:**
- $x = (y, u) \in X$
- $w = (z, v) \in X$
Saddle-point formulation: applying Brezzi theory

- the optimal control problem

\[
\min_{x \in X} J(x; \mu) \quad \text{subject to} \quad B(x, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q.
\]

has a unique solution \( x = (y, u) \in X \) for any \( \mu \in D \)

- that solution can be determined by solving the optimality system

\[
\begin{cases}
A(x(\mu), w; \mu) + B(w, p(\mu); \mu) = \langle F(\mu), w \rangle \quad \forall w \in X, \\
B(x(\mu), q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q,
\end{cases}
\]

Compact form

Given \( \mu \in D \), find \( U(\mu) \in \mathcal{X} \) s.t:

\[
\mathcal{X} = X \times Q, \quad U = (x, p), \quad W = (w, q)
\]

\[
B(U, W; \mu) = A(x, w; \mu) + B(w, p; \mu) + B(x, q; \mu)
\]

\[
F(W; \mu) = \langle F(\mu), w \rangle + \langle G(\mu), q \rangle
\]

- at this point we may apply the Galerkin-FE approximation
Optimize - then - discretize

$\mu$-OCP, optimality system

$$\text{Pb}(\mu; U(\mu))$$

$$U(\mu) \in \mathcal{X} : B(U(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}$$

Truth approximation (FEM)

$$\text{Pb}_N(\mu; U^N(\mu))$$

$$U^N(\mu) \in \mathcal{X}^N : B(U^N(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}^N$$

Main ingredients

In order to develop the Reduced Basis (RB) method:

- we firstly recast the problem in the framework of saddle-point problem [Gunzburger, Bochev, 2004]
- we then apply the well-known Brezzi theory [Brezzi, Fortin, 1991]

$$\text{existence, uniqueness, stability and optimality conditions}$$

This way we can exploit the analogies with the already developed theory of RB method for Stokes-type problems [Rozza, Veroy, 07; Rozza, Huynh, Manzoni 10]

The usual ingredients of the RB methodology are provided:

- Galerkin projection onto a low-dimensional space of basis functions properly selected by a greedy algorithm;
- an affine parametric dependence enabling to perform competitive Offline-Online splitting in the computational procedure;
- an efficient and rigorous a posteriori error estimation on the state, control and adjoint variables as well as on the cost functional.
Optimize - then - discretize - then - **reduce** approach

$\mu$-OCP, optimality system

\[ \text{Pb}(\mu; U(\mu)) \]

$U(\mu) \in \mathcal{X} : \quad B(U(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X} $ 

**Truth approximation (FEM)**

\[ \text{Pb}_N(\mu; U_N(\mu)) \]

$U_N(\mu) \in X_N : \quad B(U_N(\mu), W; \mu) = F(W) \quad \forall W \in X_N $ 

**Sampling (Greedy)**

**Space Construction**

(Hierarchical Lagrange basis)

**OFFLINE**

\[ S_N = \{ \mu^i, \quad i = 1, \ldots, N \} \]

\[ X_N = \text{span}\{ U_N(\mu^i), \quad i = 1, \ldots, N \} \]

\[ \text{dim}(X_N) = N \ll \mathcal{N} = \text{dim}(X^\mathcal{N}) \]

**Reduced Basis (RB) approximation**

\[ \text{Pb}_N(\mu; U_N(\mu)) \]

Galerkin projection

**ONLINE**

\[ U_N(\mu) \in X_N : \quad B(U_N(\mu), W; \mu) = F(W) \quad \forall W \in X_N \]

[Patera, Rozza 2006] [Rozza *et al.*, 2008] (review)
Reduced Basis Method: approximation stability

**Reduced Basis (RB) approximation:** given $\mu \in D$, find $(x_N(\mu), p_N(\mu)) \in X_N \times Q_N$:

$$
\begin{align*}
A(x_N(\mu), w; \mu) + B(w, p_N(\mu); \mu) &= \langle F(\mu), w \rangle \quad \forall w \in X_N \\
B(x_N(\mu), q; \mu) &= \langle G(\mu), q \rangle \quad \forall q \in Q_N
\end{align*}
$$

(*)

How to define the reduced basis spaces?
Reduced Basis Method: approximation stability

Reduced Basis (RB) approximation: given $\mu \in \mathcal{D}$, find $(x_N(\mu), p_N(\mu)) \in X_N \times Q_N$:

$$
\begin{align*}
\begin{cases}
    A(x_N(\mu), w; \mu) + B(w, p_N(\mu); \mu) & = \langle F(\mu), w \rangle & \forall w \in X_N \\
    B(x_N(\mu), q; \mu) & = \langle G(\mu), q \rangle & \forall q \in Q_N
\end{cases}
\end{align*}
$$

(*)

How to define the reduced basis spaces? we have to provide a spaces pair $\{X_N, Q_N\}$ that guarantee the fulfillment of an equivalent parametrized Brezzi inf-sup condition [Negri et al., 2012]

$$
\beta_N(\mu) = \inf_{q \in Q_N} \sup_{w \in X_N} \frac{B(w, q; \mu)}{\|w\|_{X} \|q\|_{Q}} \geq \beta_0, \quad \forall \mu \in \mathcal{D}.
$$

For the state and adjoint variables: aggregated spaces

$$
Y_N \equiv Q_N = \text{span}\{y_N^n(\mu^n), p_N^n(\mu^n)\}_{n=1}^N
$$

For the control variable:

$$
W_N = \text{span}\{u_N^n(\mu^n)\}_{n=1}^N
$$

Let $X_N = Y_N \times W_N$, we can prove that

- $\beta_N(\mu) \geq \alpha_N^N(\mu) > 0$ being $\alpha_N^N(\mu)$ the coercivity constant associated to the FE approximation of the PDE operator

- Brezzi theorem $\implies$ for any $\mu \in \mathcal{D}$, the RB approximation (*) has a unique solution depending continuously on the data
RB method: Offline/Online decomposition

Algebraic formulation:

\[
\begin{pmatrix}
A_N(\mu) & B_N^T(\mu) \\
B_N(\mu) & 0
\end{pmatrix}
\begin{pmatrix}
x_N(\mu) \\
p_N(\mu)
\end{pmatrix}
= \begin{pmatrix}
F_N(\mu) \\
G_N(\mu)
\end{pmatrix}
\]

affine decomposition:

\[
K_N(\mu) = \sum_{q=1}^{Q_b} \Theta^q_b(\mu) K^q_N
\]

\[
F_N(\mu) = \sum_{q=1}^{Q_f} \Theta^q_f(\mu) F^q_N
\]

\[
\sum_{q=1}^{Q_b} \Theta^q_b(\mu) K^q_N U_N(\mu) = \sum_{q=1}^{Q_f} \Theta^q_f(\mu) F^q_N
\]

- **Offline** pre-processing: compute and store the basis functions \( \{ \zeta_i, 1 \leq i \leq 5N \} \), store the matrices \( K^q_N \) and the vectors \( F^q_N \)

**Operation count:** depends on \( N, Q_b, Q_f \) and \( \mathcal{N} \)

- **Online:** evaluate coefficients \( \Theta^q_*(\mu) \), assemble the matrix \( K_N(\mu) \) and the vector \( F_N(\mu) \) and solve the reduced system of dimension \( 5N \times 5N \)

**Operation count:** \( O((5N)^3 + Q_b N^2 + Q_F N) \) independent of \( \mathcal{N}, N \ll \mathcal{N} \)
Goal: provide rigorous, sharp and inexpensive estimators for the error on the solution variables and for the error on the cost functional

A posteriori error estimation on the solution variables

\[ \| U^N(\mu) - U_N(\mu) \|_{\mathcal{X}} \leq \frac{\| r(\cdot; \mu) \|_{\mathcal{X}'} }{\hat{\beta}_{LB}(\mu)} : = \Delta_N(\mu) \]

- \( 0 < \hat{\beta}_{LB}(\mu) \leq \hat{\beta}^N(\mu) \) is a constructible lower bound of the Babuška inf-sup constant

\[ \hat{\beta}(\mu) = \inf_{W \in \mathcal{X}} \sup_{U \in \mathcal{X}} \frac{B(U, W; \mu)}{\| U \|_{\mathcal{X}} \| W \|_{\mathcal{X}}} \geq \hat{\beta}_0, \quad \forall \mu \in \mathcal{D} \]

given by the successive constraint method (SCM) (or by an interpolant surrogate);

Offline/Online strategy

- residual of the optimality system: \( r(W; \mu) = F(W; \mu) - B(U_N, W; \mu) \); we can provide the standard Offline/Online stratagem for the efficient computation of \( \| r(\cdot; \mu) \|_{\mathcal{X}'} \);

A posteriori error estimation on the cost functional

\[ | J^N(\mu) - \mathcal{J}_N(\mu) | \leq \frac{1}{2} \| r(\cdot; \mu) \|_{\mathcal{X}'} \| U^N(\mu) - U_N(\mu) \|_{\mathcal{X}} \leq \frac{1}{2} \frac{\| r(\cdot; \mu) \|_{\mathcal{X}'}^2}{\hat{\beta}_{LB}(\mu)} : = \Delta^J_N(\mu). \]
RB Method: the “complete game”

- **Offline stage** involves precomputation of FE structures required for the RB space construction and the certified error estimates.

- **Online stage** has complexity only depending on $N$ and allows resolution of the Optimal Control Problem for any $\mu \in \mathcal{D}$ with a certified error bound.

Implementation in MATLAB using MLife and rbMIT libraries.
Computational reduction

\[ \delta T^{\mathcal{N}} \rightarrow \text{FE marginal cost} \]

\[ \text{input } \mu \quad \rightarrow \quad \mu\text{-PDE} \quad \begin{cases} U^{\mathcal{N}}(\mu) & \text{FE solution} \\ J^{\mathcal{N}}(\mu) & \text{FE output} \end{cases} \]

\[ \delta T_{\mathcal{N}} \rightarrow \text{RB marginal cost} \]

\[ \text{OFFLINE} \quad \sim \\ \text{ONLINE} \]

\[ \text{input } \mu \quad \rightarrow \quad \mu\text{-PDE} \quad \begin{cases} U_{\mathcal{N}}(\mu) & \text{RB solution} \approx U^{\mathcal{N}}(\mu) \\ J_{\mathcal{N}}(\mu) & \text{RB output} \approx J^{\mathcal{N}}(\mu) \end{cases} \]

http://augustine.mit.edu
Geometrical Parametrization

✓ RB framework requires a geometrical map $T(\cdot; \mu) : \Omega \rightarrow \Omega_o(\mu)$ in order to combine discretized solutions for the space construction.

✓ This procedure enables to avoid shape deformation and remeshing (that, e.g. normally occur at each step of an iterative optimization procedure).

✓ Reduction in the complexity of parametrization: versatility, low-dimensionality, automatic generation of maps, capability to represent realistic configurations, ...

Left: Different carotid bifurcation specimens obtained by autopsy (adults aged 30-75); picture taken from Z. Ding et al., Journal of Biomechanics 34 (2001), 1555-1562.
Right: Different carotid bifurcation obtained through radial basis functions techniques.
Shape Parametrization Techniques

**Cartesian geometries:**
- Affine/nonaffine mapping “by hands”

**Complex realistic geometries:**
- Automatic affine transformation (DD) rbMIT
- Free-shape nonaffine transformations based on control points (e.g. Free-Form Deformation [Sederberg & Parry], Radial Basis Functions [Bookstein, Buhmann])
- Transfinite Mappings [Gordon, Hall]
Free-Form Deformation (FFD) Techniques

Construction:

- Parametric map: $T(x, \mu) = \sum_{l=0}^{L} \sum_{m=0}^{M} b_{l,m}^{L,M}(\Psi(x))(P_{l,m} + \mu_{l,m})$ where
  
  $$b_{\ell,m}^{L,M}(s, t) = b_{\ell}^{L}(s)b_{m}^{M}(t) = \binom{L}{\ell}(M_{m})_{m}(1 - s)^{L - \ell}s^{\ell}(1 - t)^{M - m}t^{m}$$
  
  are tensor products of Bernstein basis polynomials

- FFD mapping defined as $\Omega_{o}(\mu) = \Psi^{-1} \circ \hat{T} \circ \Psi(\Omega; \mu) =: T(\Omega; \mu)$

- Parameters $\mu_{1}, \ldots, \mu_{P}$ are displacements of selected control points
L0 - Boundary control for a Graetz convection-diffusion problem

We consider the following optimal control problem:

\[
\begin{align*}
\text{minimize} & \quad J(y_o(\mu), u_o(\mu); \mu) = \frac{1}{2} \| y_o(\mu) - y_d(\mu) \|^2_{L^2(\hat{\Omega}_o)} + \frac{\alpha}{2} \| u_o(\mu) \|^2_{L^2(\Gamma_o^C)} \\
\text{s.t.} & \quad \begin{cases}
- \frac{1}{\mu_1} \Delta y_o(\mu) + x_{o2}(1 - x_{o2}) \frac{\partial y_o(\mu)}{\partial x_{o1}} = 0 & \text{in } \Omega_o(\mu) \\
y_o(\mu) = 1 & \text{on } \Gamma_o^D \\
\frac{1}{\mu_1} \nabla y_o(\mu) \cdot n = u_o(\mu) & \text{on } \Gamma_o^C(\mu) \\
\frac{1}{\mu_1} \nabla y_o(\mu) \cdot n = 0 & \text{on } \Gamma_o^N(\mu),
\end{cases}
\end{align*}
\]

- the problem is mapped to a reference domain \( \Omega = \Omega_o(\mu_{\text{ref}}) \) with \( \mu_{\text{ref}} = (\cdot, 1, \cdot) \)
- we obtain an affine decomposition with \( Q_B = 6, Q_F = 5 \)
Boundary control for a Graetz convection-diffusion problem

Representative solution for $\mu = (12, 2, 2.5)$

Optimal control $u_N$ on $\Gamma_C$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of FE dof $N$</td>
<td>8915</td>
</tr>
<tr>
<td>Number of parameters $P$</td>
<td>3</td>
</tr>
<tr>
<td>Number of RB functions $N$</td>
<td>39</td>
</tr>
<tr>
<td>Dimension of RB linear system</td>
<td>$39 \cdot 5$</td>
</tr>
<tr>
<td>Affine operator components $Q$</td>
<td>6</td>
</tr>
</tbody>
</table>

Linear system dimension reduction 50:1
FE evaluation $t_{FE}$ (s) 14.5
RB evaluation $t_{RB}^{\text{online}}$ (s) 0.1
RB evaluation $t_{RB}^{\text{offline}}$ (s) 3970

Error estimation (○) and true error (●) for the solution (left) and the cost functional (right)
Towards reduced data reconstruction/assimilation

Sectional axial flow profile (top) and vorticity (bottom) and salient locations along a bend.

Picture taken from D. Doorly and S. Sherwin, Geometry and flow,
In Cardiovascular Mathematics, L. Formaggia, A. Quarteroni and A. Veneziani (Eds.)
L1 - Reduced data reconstruction/assimilation

- goal: to reconstruct, from areal data provided by eco-dopplers measurements, the blood velocity field in a section of a carotid artery
- surface estimation starting from scattered data: the reconstruction should take into account the shape of the domain and preserve the no-slip condition

Surface estimation problem [Azzimonti et al., 2011]

\[
\min_{y, u} J(y, u; \mu) = \sum_{i=1}^{m} \int_{\Omega_{obs,i}} |y(\mu) - z_i|^2 d\Omega + \frac{\alpha}{2} \|u(\mu)\|_{L^2}^2
\]

s.t. \(\begin{cases} -\Delta y(\mu) = u(\mu) & \text{in } \Omega(\mu_g) \\ y(\mu) = 0 & \text{on } \partial \Omega(\mu_g) \end{cases}\)

- Geometrical parametrization: Free Form Deformation
  \(P = 4\) displacements of the control points \(\bullet \bullet \bullet \),
  \(\mu_g \in (-0.15, 0.15)^4\) [Manzoni, Phd thesis]
- Parametrized observation values: \(\mu_{obs}^i = z_i, 1 \leq i \leq m = 5\)
L1 - Reduced data reconstruction/assimilation [Rozza et al., 2012, ECCOMAS]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of FE dof $N$</td>
<td>$3.3 \cdot 10^4$</td>
</tr>
<tr>
<td>Regularization parameter $\alpha$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Number of parameters $P$</td>
<td>$4 + 5$</td>
</tr>
<tr>
<td>Number of RB functions $N$</td>
<td>42</td>
</tr>
<tr>
<td>Affine components $Q_B$</td>
<td>53</td>
</tr>
<tr>
<td>Linear system dimension red.</td>
<td>160:1</td>
</tr>
<tr>
<td>RB solution $t_{RB}^{online}(s)$</td>
<td>0.013</td>
</tr>
<tr>
<td>RB certification $t_{\Delta}^{online}(s)$</td>
<td>0.98</td>
</tr>
</tbody>
</table>

To fulfill the affine parametric dependence assumption we rely on the **Empirical Interpolation Method** [Barrault et al, 2004]

Example of reconstructed profiles given different sets of (virtual) observation values:
Stokes constraint: how to extend the method

\[
\text{minimize } J(v, \pi, u; \mu) = \frac{1}{2} m(v - v_d(\mu), v - v_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu) \quad \text{subject to}
\]
\[
\begin{align*}
a(v, \xi; \mu) + b(\xi, \pi; \mu) &= \langle F(\mu), \xi \rangle + c(u, \xi; \mu) \quad \forall \xi \in V, \\
b(v, \tau; \mu) &= \langle G(\mu), \tau \rangle \quad \forall \tau \in M,
\end{align*}
\]

**Functional setting:**

\[
V = [H^1(\Omega)]^2 \quad M = L^2(\Omega) \quad \text{velocity and pressure spaces}
\]
\[
Y = V \times M \text{ state space,} \quad Q \equiv Y \text{ adjoint space,} \quad U \text{ control space}
\]

- two nested saddle-point
  - outer: optimal control
  - inner: Stokes constraint

- reduced basis functions computed by solving \( N \) times the FE approximation (with stable spaces pair for velocity and pressure variables)

- stability of the RB approximation of the Stokes constraint fulfilled by introducing suitable **supremizer operators** [Rozza & Veroy, 2007; Rozza et al., n.d.]

- stability of the RB approximation of the whole optimal control problem fulfilled by defining suitable **aggregated spaces** for the state and adjoint variables [Negri et al., 2013]
Stokes constraint: how to extend the method

minimize $J(v, \pi, u; \mu) = \frac{1}{2} m(v - v_d(\mu), v - v_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu)$ subject to

$$\begin{aligned}
a(v, \xi; \mu) + b(\xi, \pi; \mu) &= \langle F(\mu), \xi \rangle + c(u, \xi; \mu) \quad \forall \xi \in V, \\
b(v, \tau; \mu) &= \langle G(\mu), \tau \rangle \quad \forall \tau \in M,
\end{aligned}$$

Functional setting: $V = [H^1(\Omega)]^2$ $M = L^2(\Omega)$ velocity and pressure spaces
$Y = V \times M$ state space, $Q \equiv Y$ adjoint space, $U$ control space

Reminder: enrichment by supremizers operators for the Stokes equations

$$M_N = \text{span}\{\pi^N(\mu^n), \ n = 1, \ldots, N\}, \quad \text{pressure}$$
$$V_N^\mu = \text{span}\{v^N(\mu^n), T^\mu(\pi^N(\mu^n)), \ n = 1, \ldots, N\}, \quad \text{velocity}$$

being $T^\mu : M \to V$ the supremizer operator s.t.

$$(T^\mu q, w)_V = b(q, w; \mu) \quad \forall \ w \in V,$$

so that $\{V_N^\mu, M_N\}$ fulfill an equivalent RB Brezzi inf-sup stability condition [R., Veroy, et al.]
L2 - Vorticity minimization on the downstream portion of a bluff body

**GOAL:** minimize the vorticity in the wake of the body through suction/injection of fluid on the control boundary $\Gamma_C$

The state velocity and pressure variables \( \{v, \pi\} \) satisfy the Stokes equations in $\Omega(\mu_1)$ with the following boundary conditions:

\[
\begin{align*}
  v &= 0 & \text{on } \Gamma_D(\mu_1), \\
  v &= g(\mu_2) & \text{on } \Gamma_{in}, \\
  -\pi n + \nu \nabla v n &= 0 & \text{on } \Gamma_{out}(\mu_1), \\
  v_1 &= 0 & \text{on } \Gamma_C, \\
  v_2 &= u & \text{on } \Gamma_C,
\end{align*}
\]

where $g(\mu_2)$ is a parabolic inflow profile with peak velocity equal to $\mu_2$.

The cost functional is given by:

\[
\mathcal{J}(v(\mu), u(\mu); \mu) = \frac{1}{2} \int_{\Omega_{obs}} |\nabla \times v(\mu)|^2 \, d\Omega + \frac{\mu_3}{2} \|u(\mu)\|_{H^1(\Gamma_C)}^2
\]
L2 - Vorticity minimization on the downstream portion of a bluff body

\[ \mu_1 \in [0.1, 0.3] \quad \mu_2 \in [0.5, 2] \quad \mu_3^{-1} \in [1, 200] \]

<table>
<thead>
<tr>
<th>Number of FE dof ( \mathcal{N} )</th>
<th>( 3.6 \cdot 10^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of parameters ( P )</td>
<td>3</td>
</tr>
<tr>
<td>Number of RB functions ( N )</td>
<td>19</td>
</tr>
<tr>
<td>Dimension of RB linear system</td>
<td>( 19 \cdot 13 )</td>
</tr>
<tr>
<td>Affine operator components ( Q )</td>
<td>14</td>
</tr>
</tbody>
</table>

Linear system dim reduction \( 150:1 \)

FE evaluation \( t_{FE} \) (s) \( \approx 15 \)

RB evaluation \( t_{RB}^{\text{online}} \) (s) \( 0.1 \)

Average computed error and bound between the truth FE solution and the RB approximation.

Stability factor: Babuška inf-sup w.r.t. to \( \mu_1 \)
L3 - An (idealized) application in haemodynamics: a data assimilation problem

- we consider an inverse boundary problem in hemodynamics, inspired by the work [D’Elia et. al, 2011]
- parametrized geometrical model of an arterial bifurcation (with FFD)
- we suppose to have a measured velocity profile on the red section, but not the Neumann flux on $\Gamma_C$ that will be our control variable
- starting from the velocity measures we want to find the control variable in order to retrieve the velocity and pressure fields in the whole domain.

Given new geometrical configuration ($\mu_g$) and parametrized measurements $\mu_{obs}$ on the red section, we want to find the unknown Neumann boundary condition on $\Gamma_C$ and retrieve the whole velocity and pressure fields.
An (idealized) application in haemodynamics: a data assimilation problem

The state velocity and pressure variables \( \{v, \pi\} \) satisfy the following Stokes problem in \( \Omega(\mu) \):

\[
-\nu \Delta v + \nabla \pi = 0 \quad \text{in} \quad \Omega(\mu_g),
\]
\[
\text{div} \, v = 0 \quad \text{in} \quad \Omega(\mu_g),
\]
\[
v = 0 \quad \text{on} \quad \Gamma_D(\mu_g),
\]
\[
-\pi n + \nu \frac{\partial v}{\partial n} = u \quad \text{on} \quad \Gamma_C(\mu_g),
\]

where \( g(\mu_{in}) \) is a parabolic inflow profile.

Then we consider the following parametrized cost functional to be minimized

\[
\mathcal{J}(v, \pi, u; \mu) = \frac{1}{2} \int_{\Gamma_{obs}} |v - v_d(\mu_{obs})|^2 \, d\Gamma + \text{regularization}(u)
\]
L3 - An (idealized) application in haemodynamics: a data assimilation problem

Average computed error and bound between the truth FE solution and the RB approximation.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of FE dof $N$</td>
<td>$4 \cdot 10^4$</td>
</tr>
<tr>
<td>Number of parameters $P$</td>
<td>3</td>
</tr>
<tr>
<td>Number of RB functions $N$</td>
<td>17</td>
</tr>
<tr>
<td>Dimension of RB linear system</td>
<td>$17 \cdot 13$</td>
</tr>
<tr>
<td>Affine operator components $Q$</td>
<td>20</td>
</tr>
<tr>
<td>FE evaluation $t_{FE} \text{ (s)}$</td>
<td>$\approx 20$</td>
</tr>
<tr>
<td>RB evaluation $t_{RB}^{\text{online}} \text{ (s)}$</td>
<td>0.15</td>
</tr>
</tbody>
</table>
Boundary control of Navier-Stokes flow

Find \((v, \pi, \mu)\) such that the cost functional

\[
J(v, \pi, u; \mu) = F(v, \pi; \mu) + G(u; \mu)
\]

is minimized subject to the steady Navier-Stokes equations:

\[
-\nu \Delta v + (v \cdot \nabla) v + \nabla \pi = f \quad \text{in } \Omega(\mu)
\]

\[
\text{div } v = 0 \quad \text{in } \Omega(\mu)
\]

\[
v = u \quad \text{on } \Gamma_C(\mu)
\]

\[
v = 0 \quad \text{on } \Gamma_D(\mu)
\]

\[
-\pi n + \nu \nabla v \cdot n = 0 \quad \text{on } \Gamma_N(\mu).
\]

Possible choices for \(F\), viscous energy dissipation or velocity tracking type functionals:

\[
F(v, \pi; \mu) = \frac{\nu}{2} \int_{\Omega(\mu)} |\nabla v|^2 \, d\Omega,
\]

\[
F(v, \pi; \mu) = \frac{1}{2} \int_{\Omega_{\text{obs}}(\mu)} |v - v_d(\mu)|^2 \, d\Omega
\]

Regularization contribute:

\[
G(u; \mu) = \frac{\alpha}{2} \int_{\Gamma_C(\mu)} (|\nabla u|^2 + |u|^2) \, d\Gamma
\]

[Gunzburger et al., 1991], [Hou & Ravindran, 1999], [Biros & Ghattas, 1999, 2005]
Boundary control of Navier-Stokes flow

Find \((v, \pi, \mu)\) such that the cost functional

\[
\mathcal{J}(v, \pi, u; \mu) = \mathcal{F}(v, \pi; \mu) + \mathcal{G}(u; \mu)
\]

is minimized subject to the steady Navier-Stokes equations:

\[
-\nu \Delta v + (v \cdot \nabla)v + \nabla \pi = f \quad \text{in } \Omega(\mu)
\]

\[
\text{div} v = 0 \quad \text{in } \Omega(\mu)
\]

\[
v = u \quad \text{on } \Gamma_C(\mu)
\]

\[
v = 0 \quad \text{on } \Gamma_D(\mu)
\]

\[
-\pi n + \nu \nabla v \cdot n = 0 \quad \text{on } \Gamma_N(\mu).
\]

Possible choices for \(\mathcal{F}\), viscous energy dissipation or velocity tracking type functionals:

\[
\mathcal{F}(v, \pi; \mu) = \frac{\nu}{2} \int_{\Omega(\mu)} |\nabla v|^2 \, d\Omega,
\]

\[
\mathcal{F}(v, \pi; \mu) = \frac{1}{2} \int_{\Omega_{\text{obs}}(\mu)} |v - v_d(\mu)|^2 \, d\Omega
\]

Regularization contribute:

\[
\mathcal{G}(u; \mu) = \frac{\alpha}{2} \int_{\Gamma_C(\mu)} (|\nabla u|^2 + |u|^2) \, d\Gamma
\]

[Gunzburger et al., 1991], [Hou & Ravindran, 1999], [Biros & Ghattas, 1999, 2005]
Boundary control of Navier-Stokes flow: optimality system \textit{quadratic nonlinearity}

<table>
<thead>
<tr>
<th>State equation</th>
<th>Adjoint equation</th>
</tr>
</thead>
</table>
| $-\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi = \mathbf{f}$<br>$\text{div} \mathbf{v} = 0$
| $\mathbf{v} = \mathbf{u}$ on $\Gamma_C$ + other BCs | $-\nu \Delta \lambda + (\nabla \mathbf{v})^T \lambda - (\mathbf{v} \cdot \nabla)\lambda + \nabla \eta = \nu \Delta \mathbf{v}$<br>$\text{div} \lambda = 0$
| | $\lambda = 0$ on $\Gamma_C$ + other BCs |

<table>
<thead>
<tr>
<th>Optimality equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\alpha (\Delta_{\Gamma_C} \mathbf{u} + \mathbf{u}) = \eta \mathbf{n} - \nu (\nabla \lambda + \nabla \mathbf{v}) \cdot \mathbf{n}$&lt;br&gt;on $\Gamma_C$</td>
</tr>
</tbody>
</table>
Boundary control of Navier-Stokes flow: optimality system \textit{quadratic nonlinearity}

<table>
<thead>
<tr>
<th>State equation</th>
<th>Adjoint equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\nu \Delta v + (v \cdot \nabla) v + \nabla \pi = f)</td>
<td>(-\nu \Delta \lambda + (\nabla v)^T \lambda - (v \cdot \nabla) \lambda + \nabla \eta = \nu \Delta v)</td>
</tr>
<tr>
<td>(\text{div } v = 0)</td>
<td>(\text{div } \lambda = 0)</td>
</tr>
<tr>
<td>(v = u) on (\Gamma_C) + other BCs</td>
<td>(\lambda = 0) on (\Gamma_C) + other BCs</td>
</tr>
</tbody>
</table>

Optimality equation

\[-\alpha (\Delta_{\Gamma_C} u + u) = \eta n - \nu (\nabla \lambda + \nabla v) \cdot n \quad \text{on } \Gamma_C\]

- **Variational formulation**: find \(U = (v, \pi; u; \lambda, \eta) \in \mathcal{X}\) s.t.

\[G(U, W; \mu) = 0 \quad \forall W \in \mathcal{X},\]

- **Newton method**: for \(k = 1, 2, \ldots\)

\[dG[U^k](U^{k+1}, W; \mu) = -G(U^k, W; \mu) \quad \forall W \in \mathcal{X}\]

where \(dG[U](V, W; \mu)\) denotes the Fréchet derivative of \(G(\cdot, \cdot; \mu)\)
RB approximation and BRR error bound

As in the Stokes case:
- reduced basis functions computed by solving $N$ times the FE approximation
- stability of the RB approximation: \textit{supremizer operators} + \textit{aggregated spaces} for the state and adjoint variables

Nonlinear ingredients:
- Galerkin projection on $\mathcal{X}_N$ + Newton method: for $k = 1, 2, \ldots$ until convergence
  \[
dG[U^k_N](U^{k+1}_N, W_N; \mu) = -G(U^k_N, W_N; \mu) \quad \forall W_N \in \mathcal{X}_N
\]
- Brezzi-Rappaz-Raviart error bound: if
  \[
  \tau_N(\mu) = 4 \frac{\gamma(\mu) \varepsilon_N(\mu)}{\hat{\beta}^2(\mu)} < 1 \quad \text{where} \quad \varepsilon_N(\mu) = \|G(U_N, \cdot; \mu)\|_{\mathcal{X}_N'}
  \]
  then
  \[
  \|U^N(\mu) - U_N(\mu)\|_\mathcal{X} \leq \Delta_N(\mu) := \frac{\hat{\beta}(\mu)}{2\gamma(\mu)} \left(1 - \sqrt{1 - \tau_N(\mu)}\right)
  \]
**NL1 - Vorticity minimization on the downstream portion of a bluff body**

**GOAL**: minimize the vorticity in the wake of the body through suction/injection of fluid on the control boundary $\Gamma_C$

$$\mu_1^{-1} \in [5, 80] \quad \mu_2 \in [10, 60]$$

The geometry is fixed. The parameters are the regularization constant $\mu_1$ in the functional (tuning the size of the control) and the Reynolds number $\mu_2$.

minimize $J(v, u; \mu) = \frac{1}{2} \int_{\Omega_{obs}} |\nabla \times v|^2 \, d\Omega + \frac{\mu_1}{2} ||u||^2_{H^1(\Gamma_C)}$

s.t. \[
\begin{align*}
-\frac{1}{\mu_2} \Delta v + (v \cdot \nabla) v + \nabla \pi &= 0 & \text{in } \Omega \\
\text{div } v &= 0 & \text{in } \Omega \\
v &= u & \text{on } \Gamma_C
\end{align*}
+ \text{other boundary conditions}
\]
NL1 - Vorticity minimization on the downstream portion of a bluff body

Results: no greedy algorithm (due to computational limitations), computation of reduced basis in randomly chosen parameter points.

Sharpness of the error bounds depends on Reynolds number through $\hat{\beta}(\mu)$:
NL1 - Vorticity minimization on the downstream portion of a bluff body

<table>
<thead>
<tr>
<th>FE evaluation $t_{FE}$ (s)</th>
<th>$\approx 60$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RB evaluation $t_{RB}^{\text{online}}$ (s)</td>
<td>0.9</td>
</tr>
<tr>
<td>Number of RB functions $N$</td>
<td>35</td>
</tr>
</tbody>
</table>

Uncontrolled solution

\[
\mu = \left[ \frac{1}{10}, 45 \right] \quad \mu = \left[ \frac{1}{55}, 30 \right] \quad \mu = \left[ \frac{1}{80}, 45 \right]
\]
we provided a **certified** RB method for the efficient solution of parametrized optimal control problems with **high dimensional** control variable

- reduction to low-dimensionality of the whole control problem and not just of the state equation

- **key ingredient**: saddle-point formulation of the optimal control problem

- full Offline/Online decomposition strategy

- ...
In the context of linear-quadratic problems, possible developments guidelines are related to:

- the study of the **time-dependent** case, with a suitable POD approach, extending the work in [Dedé, 2010, 2011]. While from the theoretical point of view the non-stationary case does not represent a challenging task, from the computational point of view it requires very efficient numerical methods;

- to add **control and/or state constrains** the first case could be treated straightforwardly while the second would require a more involved analysis both from the theoretical and algorithmic point of view;

- further applications based on **non-linear state equation**: in particular optimal flow control of Navier-Stokes equations in haemodynamics.
Conclusions and perspectives

Data reconstruction/assimilation
e.g. boundary data for blood flow simulations

Real-time context
Output evaluation

Shape reconstruction
from patient-dependent configurations

Many-query context
Reduced FSI problems

Shape optimization/design of cardiovascular devices


