One-zone Arnett Model of Radioactive Transients

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Introduction

Here we construct a simple, one-zone analytic model for the evolution of a radioactive cloud of gas. This will allow us to (very approximately) calculate the light curves of neutron star merger outflows (along with supernovae and all kinds of other transients). Our discussion follows a simplified version of the more detailed analytic model for supernova light curves presented in the classic papers of Arnett (Arnett 1980 , Arnett 1982).

Consider a cloud of gas with mass M ejected from a merger. We'll assume that, initially, the total kinetic energy and internal energy of the cloud have a roughly equal value, E_0. The fundamental equation describing the evolution of the expanding cloud is the first law of thermodynamics, which expresses energy conservation

$$\frac{dE(t)}{dt} = -p\frac{dV}{dt} + \dot{\epsilon} - L(t) \tag{1}$$

where $E(t)$ is the internal energy, $p(t)$ the pressure of the remnant, $\dot{\epsilon}$ is the heating rate (ergs s^{-1}) due to radioactive decay (or some other sources) and $L(t)$ is the luminosity (ergs s^{-1}) escaping the cloud (i.e., the light curve) which is what we want to figure out.

Equation 1 takes into the account energy loses due to expansion and radiation, and heating due to radioactivity. We can solve this equation relatively easily, provided we make 5 simplifying assumptions.

assumption 1: We consider a spherical, one-zone model of the cloud. That is, we take the density, ρ, and temperature, T, to be constant throughout the cloud of radius R. This is obviously a gross approximation of the actual remnant structure, but it will turn out to give insightful scaling relations.

assumption 2: The cloud radius expands freely as $R = vt$ where $v = \sqrt{2E_0/M}$ is the characteristic velocity, and t is the time since ejection. In this case (known as *homologous* expansion) the volume of the remnant increases with time as $V(t) = V_0(t/t_0)^3$, where the initial volume is $V_0 = (4\pi/3)R_0^3$. Here, R_0 is the initial scale of the system and $t_0 = R_0/v$ the initial expansion time. Homologous expansion should be a very good approximation after a several expansion times, t_0.

assumption 3: Radiation energy dominates over gas energy. In this case, the internal energy density is given by

$$u = aT^4 \tag{2}$$

while the pressure is given by

$$p = \frac{1}{3}aT^4 \tag{3}$$

It is easy to show that the assumption of radiation domination is well motivated given the temperatures and densities we will find for the cloud.

assumption 4: The radiation leaking from the cloud is given by the diffusion equation in spherical coordinates is

$$L(r) = -4\pi r^2 \frac{c}{3\kappa\rho}\frac{\partial}{\partial r}u(r) \tag{4}$$

To properly calculate the derivative requires that we know the run of the internal energy, $u(r)$ with radius. However, since we are using a one-zone model we can approximate the value of a spatial derivative as some quantity over the characteristic length scale

$$\frac{\partial u}{\partial r} \approx -\frac{u(t)}{R(t)} \tag{5}$$

assumption 5: The opacity, κ, is a constant (i.e., unchanging is time and space). For ordinary supernova material, a reasonable value is $\kappa \approx 0.1\ \mathrm{cm}^2\ \mathrm{g}^{-1}$. For the much more opaque r-process ejecta from neutron star mergers, the value is more like $\kappa \approx 10\ \mathrm{cm}^2\ \mathrm{g}^{-1}$.

Problems

1) First, consider the case of adiabatic expansion where $L = \dot{\epsilon} = 0$ (i.e., no heat is entering or leaving the system). Show that temperature of a homologous expanding, radiation dominated cloud cools adiabatically as $T(t) = T_0(t/t_0)^{-1}$.

2) Next, consider the case where there is no heating ($\dot{\epsilon} = 0$) but some radiation escapes the remnant. Solve equation 1 for the energy density $u(t)$ as a function of time. Then use the diffusion equation[1] to derive an analytic formula for the light curve, $L(t)$. Find simple expressions (in terms of the basic physical parameters E, R_0, κ, and M) for the characteristic luminosity, L_{lc}, of the transient and the characteristic time scale, t_{lc}, on which the luminosity declines. Give rough

[1] You can continue to approximate the spatial derivative by equation 5. Note that the radius and density in this equation also evolve with time. You can replace them with $R(t) = vt$ and $\rho(t) = M/V(t)$

values of these quantities for a neutron star merger outflow with $M = 10^{-2}\ M_\odot, E_0 = 10^{50}$ ergs.

Comment: The timescale $t_{\rm lc}$ you have derived gives the effective diffusion time in a (homologously) expanding medium, which is in general useful for optically thick outflows. It is different than the familiar static diffusion time because the density (and hence optical depth) drop as the remnant expands, making it easier for photons to escape. Note that the diffusion time in a static medium can be written $t_{\rm d} \sim \kappa M/R_0 c$. You have therefore shown that $t_{\rm sn} \propto \sqrt{t_{\rm d} t_{\rm e}}$, where $t_{\rm e} \approx R_0/v$ is the the characteristic expansion time of the cloud. In other words, the time it takes photons to diffuse out of an expanding medium is given by the geometric mean of $t_{\rm d}$ and $t_{\rm e}$.

3) You'll see that the luminosity you predicted in part 2) is very dim. This is because most if the initial internal energy of the cloud is lost to expansion before it has time to be radiated away. We therefore need radioactivity to reheat the cloud at later times.

Consider now the case where $\dot{\epsilon} \neq 0$. Show that at the maximum of the light curve (i.e., $dL/dt = 0$), the luminosity of the event is equal to the instantaneous radioactive energy deposition, i.e., $L(t_{\rm peak}) = \dot{\epsilon}(t_{\rm peak})$. This is known as Arnett's law, and is very useful for estimating the amount of radioactive material present based only on the observed peak luminosity.

4) There is no simple, general solution for the case of $\dot{\epsilon} \neq 0$, but you should be able to write the luminosity L as an integral over $\dot{\epsilon}$. Consider attacking the differential equation using an integrating factor

5) An approximate expression for the radioactive decay energy rate for r-process ejecta is

$$\dot{\epsilon} \approx 1 \text{ eV s}^{-1} \left(\frac{t}{1 \text{ day}} \right)^{-1.5} \tag{6}$$

Which applies for $t > 1$ s or so. Write a code (or use an existing integration package) to do the integral found in part 4). You now have calculated your own kilonova light curve!