GIZMO: Radiation-Hydrodynamics Notes

Philip F. Hopkins[1]

[1] *TAPIR, Mailcode 350-17, California Institute of Technology, Pasadena, CA 91125, USA. E-mail:phopkins@caltech.edu*

ABSTRACT

Description of some code-details aspects of the radiation-hydrodynamics (RHD) module implementation, in the code GIZMO. Please make sure you have read the discussion in the User Guide about the compile-time flags and physics options for RHD before reading this, otherwise the options will not make sense. Also the User Guide has appropriate citation information for anyone using these modules.

Key words: methods: numerical — hydrodynamics — cosmology: theory — elementary particles — dark matter

1 EQUATIONS

The RHD implementations in GIZMO all attempt to solve the RHD equations in the usual mixed-frame approximation where the specific intensity is measured in an Eulerian frame (the "lab frame" or "simulation frame") while the radiation-matter interaction terms are computed in a frame co-moving with the fluid (Mihalas & Mihalas 1984; Lowrie et al. 1999), with the assumption that the flow is non-relativistic so we keep terms up to $\mathcal{O}(v^2/c^2)$. In what follows, *all* radiation quantities should be understood to be defined at a given specific frequency ν (e.g. $I = I_\nu$, $e_r = e_{r,\nu} = de_r/d\nu$, $\kappa = \kappa_\nu$) or (equivalently) integrated over an infinitesimally small frequency bin (so that we can make the locally-grey approximation over the width of the bin): we discuss this further below. For convenience, we will drop the subscript ν on all quantities but take it to be understood. With these terms the *gas* equations become:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + ... = \sum_\nu \left(\psi_a + \psi_s\right)\mathbf{g}_r + \frac{\tilde{\mathbf{u}}}{\tilde{c}^2}\left(\dot{e}_{\rm abs} - \dot{e}_{\rm em}\right) \quad (1)$$

$$\frac{\partial e_{\rm g}}{\partial t} + ... = \sum_\nu \frac{c}{\tilde{c}}\left(\dot{e}_{\rm abs} - \dot{e}_{\rm em}\right) + \left(\psi_s - \psi_a\right)\mathbf{u}\cdot\mathbf{g}_r \quad (2)$$

where the "..." terms refer to all the *other* (non-radiation) terms in the gas momentum and energy equations; t is time and the partial derivatives ∂ are defined in the Eulerian frame; $(\rho, \mathbf{u}, e_{\rm g})$ are the gas density, simulation-frame velocity, and total energy, $\tilde{\mathbf{u}} \equiv (\tilde{c}/c)\,\mathbf{u}$, and we define:

$$\psi_{a,s} \equiv \frac{\sigma_{a,s}}{\tilde{c}} = \frac{\rho\,\kappa_{a,s}}{\tilde{c}} \quad (3)$$

$$\dot{e}_{\rm abs} = \tilde{c}^2\,\psi_a\,e_r \quad (4)$$

$$\dot{e}_{\rm em} = \tilde{c}^2\int \psi_a\, j_\nu^{\rm em}\, d\Omega = \tilde{c}^2\,\psi_a\, 4\pi\,\langle j_\nu^{\rm em}\rangle \quad (5)$$

$$\mathbf{g}_r \equiv \mathbf{f}_r - \tilde{\mathbf{u}}\cdot(e_r\mathbb{I} + \mathbb{P}_r) \quad (6)$$

$$e_r \equiv \int I_\nu\, d\Omega = 4\pi J_\nu \quad (7)$$

$$\mathbf{f}_r \equiv \tilde{c}\int \mathbf{n} I_\nu\, d\Omega = 4\pi\,\tilde{c}\,\mathbf{H}_\nu = \frac{\tilde{c}}{c}\mathbf{F}_r \quad (8)$$

$$\mathbb{P}_r \equiv \int \mathbf{n}\otimes\mathbf{n} I_\nu\, d\Omega = 4\pi\,\mathbb{K}_\nu \equiv e_r\,\mathbb{D} \quad (9)$$

where e_r is the radiation energy density, $\mathbf{f}$ and $\mathbf{F}$ are the "effective" and "true" radiation fluxes, and $\mathbb{P}_r$ is the radiation pressure tensor (with $\mathbb{D}$ the dimensionless "Eddington tensor" and $\mathbb{I}$ the identity tensor), each defined as moments of the intensity as above ($\otimes$ is the outer product). The ψ terms are given in terms of $\sigma_{a,s}$ and $\kappa_{a,s}$, the absorption and scattering coefficients or opacities, respectively. Because the radiation terms are "per frequency," the *total* contribution to the gas momentum and energy equations should be integrated or summed over all frequencies.

The radiation intensity equation, at a specific ν, assuming the absorption and scattering coefficients are *isotropic* (for now), is:

$$\frac{1}{\tilde{c}^2}\left[\frac{\partial I_\nu}{\partial t} + \tilde{c}\,\mathbf{n}\cdot\nabla I_\nu\right] = \psi_a\left(j_\nu^{\rm em} - I_\nu\right) + \psi_s\left(J_\nu - I_\nu\right) + 3\,\mathbf{n}\cdot\boldsymbol{\beta}\,\psi_a\left(j_\nu^{\rm em} - J_\nu\right) + \mathbf{n}\cdot\boldsymbol{\beta}\left(\psi_a + \psi_s\right)\left(I_\nu + 3J_\nu\right) - 2\,\psi_s\,\boldsymbol{\beta}\cdot\mathbf{H}_\nu - \left(\psi_a - \psi_s\right)\boldsymbol{\beta}\cdot\left\{\boldsymbol{\beta}\cdot\left(J_\nu\mathbb{I} + \mathbb{K}_\nu\right)\right\} \quad (10)$$

where $\boldsymbol{\beta} \equiv \mathbf{u}/c$. Note that for convenience in these notes we absorb one factor of $\tilde{c}$ into the definition of intensity (the more common convention would be to define $I_\nu \rightarrow \tilde{c}\,I_\nu$ here) – this is just an arbitrary normalization convention chosen, as it makes the conservative quantity in I_ν (number of photons per unit volume propagating in a given direction) more obvious. Taking the first and second moments of the intensity equation, we have:

$$\frac{\partial e_r}{\partial t} + \nabla\cdot\mathbf{f} = \left(\dot{e}_{\rm em} - \dot{e}_{\rm abs}\right) + \left(\psi_a - \psi_s\right)\tilde{\mathbf{u}}\cdot\mathbf{g}_r \quad (11)$$

$$\frac{1}{\tilde{c}^2}\frac{\partial \mathbf{f}_r}{\partial t} + \nabla\cdot\mathbb{P}_r = -\left(\psi_a + \psi_s\right)\mathbf{g}_r + \frac{\tilde{\mathbf{u}}}{\tilde{c}^2}\left(\dot{e}_{\rm em} - \dot{e}_{\rm abs}\right) \quad (12)$$

In all of the above, we denote the true speed of light as c, while $\tilde{c}$ represents the (optional) "reduced" speed of light (RSOL) which can be used in the code.[1] Reducing the speed of light allows the timestep to increase proportionally.

Following Mihalas & Mihalas (1984); Stone et al. (1992), an equivalent form of the radiation moments equations in the Lagrangian frame, accurate to leading order in $\mathcal{O}(v/c)$ in all relevant (streaming/diffusion) limits is given by:

$$\rho\frac{D}{Dt}\left(\frac{e_r'}{\rho}\right) + \nabla\cdot\mathbf{f}_r' = \left(\dot{e}_{\rm em}' - \dot{e}_{\rm abs}'\right) + \mathbb{P}_r' : (\nabla\cdot\tilde{\mathbf{u}}) \quad (13)$$

$$\frac{\rho}{\tilde{c}^2}\frac{D}{Dt}\left(\frac{\mathbf{f}_r'}{\rho}\right) + \nabla\cdot\mathbb{P}_r' = -\left(\psi_a + \psi_s\right)\mathbf{f}_r' \quad (14)$$

We solve the equations for e_r and $\mathbf{f}_r$ as other quantities in the code snapshots are lab-frame, however we use this to note that we can solve Eqns. 11-12 in the comoving frame between cells in conservative form without loss of accuracy by taking:

$$\frac{1}{\tilde{c}^2}\left[\frac{\partial I_\nu}{\partial t} + \tilde{c}\,\mathbf{n}\cdot\nabla I_\nu\right] \rightarrow \frac{1}{\tilde{c}^2}\left[\rho\frac{D}{Dt}\left(\frac{I_\nu}{\rho}\right) + \tilde{c}\,\mathbf{n}\cdot\nabla I_\nu\right] - \frac{\nabla\cdot(\tilde{\mathbf{u}} I_\nu)}{c} \quad (15)$$

[1] The various terms in $\tilde{c}$ versus c are described in Skinner & Ostriker (2013). They are constructed so that, in local steady-state, the radiation energy e_r and flux $\mathbf{F}_r$, and the heating and momentum flux onto gas, all take on exactly their "true" values (those they would have with $\tilde{c} = c$), independent of $\tilde{c}$. These terms are especially important in the tight-coupling limit. We also extend the treatment in Skinner & Ostriker (2013) to one-order higher in $\mathcal{O}(v/c)$, which includes the "beaming" terms in $\tilde{\mathbf{u}}/\tilde{c}^2$. Note that the "total" momentum and energy $(e_g + e_r)$ do not appear to be naively conserved for $\tilde{c} \neq c$: this is because the photon energy injection rate is lower than the "true" value by a factor $\tilde{c}/c$, so the conserved quantity should instead follow $e_g + (c/\tilde{c})\,e_r$ in steady-state (and likewise for the gas+radiation momentum).

which allows us to re-write the conservative equations in a particularly useful form, given in Appendix A.

2 METHODS

In GIZMO, we solve the equations above with one of several methods, set by different compile-time flags as described below:

(i) `RT_LOCALRAYGRID` solves the intensity equation Eq. 10 directly. This is "exact" radiation transport. The intensity is solved on a grid in space (using the gas cells as the grid), in angle, with the number of angular bins set to $4N(N+1)$ where N is the value of `RT_LOCALRAYGRID`. The methodology is based on Jiang et al. (2014). In principle, this can accurately handle any RHD regimes. This imposes a timestep $\lesssim \Delta x/\tilde{c}$, and expense scales with number of rays.

(ii) `RT_LEBRON` solves a simplified approximate version of the intensity equation Eq. 10. The gas-radiation coupling terms are treated in an angle-and-space-averaged fashion within the kernel around the emission source and absorption source, directly ray-traced assuming optically thin transport in between (so it ignores shadowing and angular collimation of radiation, as well as diffusive limits). The methods are described in Hopkins et al. (2018, 2019). This is most useful for point sources in a mostly-optically-thin medium (by volume); it assumes an infinite speed-of-light (ray-tracing limit) and is solved in the gravity solver, imposing no new timestep constraint.

(iii) `RT_M1` solves the coupled radiation energy and flux moments equations Eq. 11-12, but not the intensity Eq. 10. This means one must assume a form of $\mathbb{P}$ or $\mathbb{D}$ to close the system. This uses the usual M1 closure (Levermore 1984):

$$\mathbb{D} \rightarrow \mathbb{D}_{\rm M1} = \left(\frac{1-\chi}{2}\right)\mathbb{I} + \left(\frac{3\chi-1}{2}\right)\hat{\mathbf{f}}\otimes\hat{\mathbf{f}} \quad (16)$$

$$\chi \equiv \frac{3+4\xi^2}{5+2\left[4-3\xi^2\right]^{1/2}} \quad , \quad \xi \equiv \frac{|\mathbf{F}_r|}{c\,e_r} \quad (17)$$

This interpolates between an advective flux in the optically thin regime and a diffusive isotropic flux in the thick regime. The scheme can handle shadowing and is computationally efficient, but the fluid approximation (closing the moments equations at second order, instead of solving for all angles $\mathbf{n}$) means certain limits, such as colliding rays, cannot be accurately handled. This imposes a timestep $\lesssim \Delta x/\tilde{c}$.

(iv) `RT_FLUXLIMITEDDIFFUSION` solves just the radiation energy equation Eq. 11, in the usual flux-limited diffusion (FLD) limit (Levermore 1984). The system is closed by assuming (1) an isotropic Eddington tensor $\mathbb{D} = \mathbb{I}/3$, and (2) that the term $\tilde{c}^{-2}\,\partial\mathbf{f}_r/\partial t \rightarrow 0$ in Eq. 12 – i.e. the flux equation has reached a local steady-state solution, giving the solution:

$$\mathbf{f}_r \rightarrow \mathbf{f}_r^{\rm FLD} = -\frac{\nabla\cdot\mathbb{P}_r}{\psi_a+\psi_s} + \tilde{\mathbf{u}}\cdot(e_r\mathbb{I}+\mathbb{P}_r) + \frac{\tilde{\mathbf{u}}\,(\dot{e}_{\rm em}-\dot{e}_{\rm abs})}{\tilde{c}^2\,(\psi_a+\psi_s)} \quad (18)$$

It is well-known that this can lead to super-luminal transport in the optically-thin limit, so to prevent this, we actually take $\mathbf{f}_r = \lambda_{\rm FLD}\,\mathbf{f}_r^{\rm FLD}$ where $\lambda_{\rm FLD}$ is a dimensionless flux-limiter with the form:

$$\lambda_{\rm FLD} \equiv \frac{3\,(2+\xi)}{6+3\xi+\xi^2} \quad , \quad \xi \equiv \frac{|\mathbf{F}_r^{\rm FLD}|}{c\,e_r} \quad (19)$$

This being a diffusion problem it imposes a timestep $\lesssim \Delta x^2\,\kappa\,\rho/\tilde{c}$. Note that some simpler FLD implementations in the literature keep just the $\nabla\cdot\mathbb{P}_r$ part of $\mathbf{f}_r$: however that would give the wrong solutions in the tightly-coupled limit (where FLD should otherwise be most accurate).

(v) `RT_OTVET` makes the identical assumptions to FLD for $\mathbf{f}_r$ and $\mathbf{f}^{\rm FLD}$ and the slope-limiter $\lambda_{\rm FLD}$, except $\mathbb{D}$ is replaced with the Eddington tensor $\mathbb{D}_{\rm thin}$ (normalized to unity trace) given by assuming an optically-thin flux ($\propto L_i\,\hat{\mathbf{r}}_{ij}/r_{ij}^2$ for source luminosity L_i at source i at distance $\mathbf{r}_{ij}$ from gas element j) from all sources as if they are point sources (akin to `RT_LEBRON`), as described in Gnedin & Abel (2001). This imposes a timestep $\lesssim \Delta x^2\,\kappa\,\rho/\tilde{c}$.

3 INPUTS IN THE CODE

To fully-determine the RHD equations in § 1, we need to specify (1) the speed of light $\tilde{c}$, (2) the band[s] or wavelength[s] being solved, (3) the corresponding emission/sources $\dot{e}_{\rm em}$ in each band, and (4) the absorption and scattering opacities $\kappa_{a,s}$.

(i) Speed-of-light: The compile-time parameter `RT_SPEEDOFLIGHT_REDUCTION` defines the dimensionless value $\tilde{c}/c$. This must be >0 and ≤ 1 or unphysical behavior will result. If not defined, we default to $\tilde{c}=c$. The usual "rule of thumb" is this must be set faster than other relevant speeds in the problem, so e.g. advection does not "outpace" light travel.

(ii) Bands: These are discussed below. GIZMO allows you to modularly mix-and-match different wavelengths and bands for the RHD, via compile-time flags (e.g. `RT_OPTICAL_NIR` and many others), or add your own. If you wish to add a new band, you must add the relevant code in a couple places: first in `allvars.h` define the band compile-time flag and which frequency "bin" it will use: follow the template of existing bands like `RT_OPTICAL_NIR`. Then you must at a minimum add code to two places in `rt_utilities.c`, as described below: the source luminosity/emissivity $\dot{e}_{\rm em}$, and the opacities $\kappa_{a,s}$ for that band.

(iii) Sources: The function `rt_get_source_luminosity` in the file `rt_utilities.c` defines the source functions/luminosities/emissivities for every waveband, for each resolution element i. The pre-built wavelengths each have their own block within this code: follow their template if you are adding a new band. Note that this handles *both* gas emissivity (the $\dot{e}_{\rm em}$ above) but also any other source of luminosity from non-gas elements in the simulation: you can add emission from point-sources like stars or compact objects (star/black hole/sink particles), annihilation (dark matter or collisionless particles), particulates or PIC particles (dust grains/cosmic rays), etc. These will be appropriately added to the radiation fields and equations-of-motion. Note that because of the finite-volume solver, you need to specify the *volume or mass-integrated luminosity* of element i, i.e. $L_{\rm em,\,i} \equiv \int_{\Omega_i}\dot{e}_{\rm em}\,d^3\mathbf{x} = \int_i(\dot{e}_{\rm em}/\rho)\,dm$. If you are using sources other than gas, be sure to set `RT_SOURCES` at compile time to allow those sources to radiate. See User Guide for custom options for source injection, etc.

(iv) Opacities: The opacities $\kappa_{a,s}$ are specified in two functions in `rt_utilities.c`. First, the *total extinction opacity* $\kappa_t \equiv \kappa_a+\kappa_s$ is specified in the function `rt_kappa`. This is an opacity (cross-section per unit mass). Second, the difference between absorption and scattering is specified in the function `rt_absorb_frac_albedo`. This specifies the dimensionless ratio $\kappa_a/\kappa_t = \kappa_a/(\kappa_a+\kappa_s) = 1-A$ (where A is the albedo), i.e. the fraction of the extinction owing to absorption. If this is not specified for a given band, the code will default to assume equal absorption and scattering opacities.

4 EXISTING BANDS & SPECIAL BEHAVIORS

A wide variety of sources are pre-coded for different modules in GIZMO. See the User Guide for up-to-date details. These include soft and hard X-rays (`RT_XRAY`); HI, HeI,

and HeII ionizing photons (RT_CHEM_PHOTOION); Lyman-Werner (H_2 dissociating) photons (RT_LYMAN_WERNER); photoelectric (dust-ionizing) photons (RT_PHOTOELECTRIC); near-ultraviolet broad-band continuum (RT_NUV); optical-near infrared broad-band continuum (RT_OPTICAL_NIR); free-free emission and absorption for a fully-ionized gas with Thompson scattering (RT_FREEFREE); and broad-band infrared dust transfer (RT_INFRARED).

Most of these bands are built for applications including star and planet and galaxy formation, accretion, black holes and active galactic nuclei, compact object dynamics, and more – so the relevant sources which are current coded will assume things like "star particles" for the sources. If you want to modify these (to e.g. include the relevant band in a stellar structure/dynamics simulation), you need to modify the source function (though the opacities may be fine already, you should obviously check, or code a new band if appropriate). Likewise if you want to include new sources (e.g. gas X-ray emission for sufficiently hot plasmas) you can simply add the relevant lines to the source functions, as described above. The free-free and infrared modules do include gas as a source.

Most of these modules have additional interactions which need to be custom coded for the appropriate physics. For example, RT_XRAY, RT_CHEM_PHOTOION, RT_LYMAN_WERNER, RT_PHOTOELECTRIC, and RT_INFRARED all interact with the gas chemistry and/or thermal cooling physics (with different cooling/chemistry modules such as default COOLING or GRACKLE or CHIMES or the rt_chemistry sub-modules using different parts of these RHD bands if they are enabled). This doesn't appear explicitly in our general RHD equations in § 1 because its really a question of what the radiation does to the gas once it is absorbed, modifying ("within" the gas equations) other equations like the internal chemistry. X-rays interact via Compton cooling, infrared and photoelectric via setting the dust temperature and photoelectric emission and hence the dust-gas interaction/cooling terms, ionizing and Lyman-werner bands directly change the ionization state of the gas. If you are adding new modules which interact with the cooling/thermodynamics/chemistry of the gas, you should start by looking at the behavior of these.

Some bands also are specifically set up to re-process radiation absorbed in other bands: e.g. if you enable RT_INFRARED and some of the other bands whose opacity is dominated by dust (e.g. RT_OPTICAL_NIR), then the code is set up so that radiation absorbed by the dust in those other bands will be assumed to re-emit in the dust band.

Most of the above bands are very narrow, so we can treat the equations above in the grey limit over the width of the band to relatively high accuracy. Some, like free-free, are broad but the opacities are very close to grey over the regime of applicability. But it is possible to account for spectral shape within a band, if needed. One example is RT_INFRARED. Here the entire IR is modeled as a modified blackbody in shape, but we do *not* assume (unlike many studies in the literature) that the dust or radiation or gas temperatures are identical. The radiation temperature, or equivalently spectral shape or median/peak wavelength is evolved including absorption and emission and advection/mixing between cells: the key assumption is that *both* photon number and energy are integrated exactly (so tracking the radiation temperature is done in a manner such that the code gives the correct mean energy per photon exactly); this is then used to compute band-integrated but spectral-shape-dependent opacities.

REFERENCES

Gnedin N. Y., Abel T., 2001, New Astronomy, 6, 437
Hopkins P. F., et al., 2018, MNRAS, 480, 800
Hopkins P. F., Grudić M. Y., Wetzel A., Kereš D., Faucher-Giguère C.-A., Ma X., Murray N., Butcher N., 2019, MNRAS, p. 2736
Jiang Y.-F., Stone J. M., Davis S. W., 2014, ApJS, 213, 7
Levermore C. D., 1984, Journal of Quantitative Spectroscopy and Radiative Transfer, 31, 149
Lowrie R. B., Morel J. E., Hittinger J. A., 1999, ApJ, 521, 432
Mihalas D., Mihalas B. W., 1984, Foundations of radiation hydrodynamics. New York, Oxford University Press, 731 p.
Skinner M. A., Ostriker E. C., 2013, ApJS, 206, 21
Stone J. M., Mihalas D., Norman M. L., 1992, ApJS, 80, 819

APPENDIX A: ALTERNATIVE FORM OF THE RSOL EQUATIONS

For purposes of understanding how the RSOL enters the dynamics, it is convenient to re-write the Eulerian equations solved in the following form:

$$\begin{aligned}\frac{\rho}{\tilde{c}}\frac{D}{Dt}\left(\frac{I_\nu}{\rho}\right) + \nabla\cdot(\mathbf{n}I_\nu - \boldsymbol{\beta}I_\nu) &= R_a\left(j_\nu^{\rm em} - I_\nu\right)\\ &+ R_s\left(J_\nu - I_\nu\right) + 3\,\mathbf{n}\cdot\boldsymbol{\beta}R_a\left(j_\nu^{\rm em} - J_\nu\right)\\ &+ \mathbf{n}\cdot\boldsymbol{\beta}\left(R_a + R_s\right)\left(I_\nu + 3J_\nu\right) - 2R_s\boldsymbol{\beta}\cdot\mathbf{H}_\nu\\ &- \left(R_a - R_s\right)\boldsymbol{\beta}\cdot\left\{\boldsymbol{\beta}\cdot\left(J_\nu\mathbb{I} + \mathbb{K}_\nu\right)\right\} \qquad \text{(A1)}\end{aligned}$$

where $R_{a,s} \equiv \rho\kappa_{a,s}$. Here, *all* variables have the identical meaning to their usual meaning with $\tilde{c} = c$, regardless of the RSOL. The *only* term where the RSOL $\tilde{c}$ appears is in the time derivative D/Dt. Note that $\rho D(X/\rho)/Dt$ is *defined* to be $\partial X/\partial t + \nabla\cdot(\mathbf{u}X)$ – this does not imply a relativistic boost, since X is not Lorentz-transformed to a comoving frame – it is simply a linear coordinate transformation convenient for our moving fluid elements. Taking moments of Eq. A1 we obtain:

$$\begin{aligned}\frac{\rho}{\tilde{c}}\frac{D}{Dt}\left(\frac{e_r}{\rho}\right) + \nabla\cdot\left(\tilde{\mathbf{F}}_r - \boldsymbol{\beta}e_r\right) &\qquad \text{(A2)}\\ = R_a\left(4\pi\langle j_\nu^{\rm em}\rangle - e_r\right) + \left(R_a - R_s\right)\boldsymbol{\beta}\cdot\mathbf{G}_r&\\ \frac{\rho}{\tilde{c}}\frac{D}{Dt}\left(\frac{\tilde{\mathbf{F}}_r}{\rho}\right) + \nabla\cdot\left(\mathbb{P}_r - \boldsymbol{\beta}\otimes\tilde{\mathbf{F}}_r\right) &\qquad \text{(A3)}\\ = -\left(R_a + R_s\right)\mathbf{G}_r + \boldsymbol{\beta}R_a\left(4\pi\langle j_\nu^{\rm em}\rangle - e_r\right)&\end{aligned}$$

where $\mathbf{G}_r \equiv \tilde{\mathbf{F}}_r - \boldsymbol{\beta}\cdot(e_r\mathbb{I} + \mathbb{P}_r)$ and $\tilde{\mathbf{F}}_r \equiv \mathbf{F}_r/c$.

These are the forms solved. The association of $\tilde{c}$ with the derivative D/Dt as opposed to e.g. $\partial/\partial t$ is arbitrary, but deliberate, as it ensures more rapid convergence with respect to $\tilde{c}/c$ in systems with relatively large velocities and maintains the Galilean invariance of tightly-coupled (diffusive) solutions. It also makes it more clear how $\tilde{c}/c$ effectively enters as a time-unit rescaling for the radiation. Finally, it also makes the ordering in $\mathcal{O}(v/c)$ more clear – for example, one can easily verify that the $\beta\otimes\tilde{\mathbf{F}}_r$ term is never important in any (streaming or static/dynamical diffusion) limits, and can be safely dropped, while the βe_r and $\beta\cdot\mathbf{G}_r$ terms are important given our Eulerian definition of e_r and $\tilde{\mathbf{F}}_r$ in the diffusive limits and must be retained (Mihalas & Mihalas 1984).