Ay 123 Lecture IV Equations of State

6.1 Electron gas

Thus far we have mainly dealt with ideal gases and radiation described by classical physics. However, in some cases, quantum mechanical effects become very important. This is true when electrons become degenerate.

Basic idea:

From Heisenberg uncertainty principle, the position and momentum of a particle are not well-defined:

$$\Delta x \, \Delta p \gtrsim \frac{\hbar}{2}$$

So, electrons (or any fermion, but not bosons) cannot be packed into too small of a space, unless their expectation value of momentum increases.

We can use the Heisenberg uncertainty principle to estimate an equation of state for degenerate electrons:

$$\Delta x \, \Delta p \simeq \frac{\hbar}{2}$$

$$\Rightarrow \underbrace{n_e^{-1/3}}_{\#\text{ density of electrons}} \; \underbrace{p_e}_{\text{momentum of electrons}} \sim \hbar$$

$$\Rightarrow p_e \sim \hbar n_e^{1/3}$$

$$\sim \hbar \left(\frac{\rho}{\mu_e m_p}\right)^{1/3}$$

Where μ_e is the baryonic mass number per electron.
$\mu_e = 1$ for hydrogen, $\mu_e = 2$ for ^4He, ^{12}C, ^{16}O.
The pressure exerted by the electrons is

$$P_e \sim n_e p_e \underbrace{V_e}_{\text{velocity}}$$

For non-relativistic electrons, $V_e = \frac{p_e}{m_e}$

$$\Rightarrow P_e \sim \frac{1}{m_e} n_e p_e^2$$

$$\sim \frac{1}{m_e}\left(\frac{\rho}{\mu_e m_p}\right)^{5/3} \hbar^2$$

$$\Rightarrow \boxed{P_e \sim \frac{\hbar^2}{(\mu_e m_p)^{5/3} m_e} \rho^{5/3}}$$

Non-relativistic electrons

This is a polytropic EOS, with $K = \frac{\hbar^2}{(\mu_e m_p)^{5/3} m_e}$, $\gamma = 5/3$

For relativistic electrons, $V_e = c$. Then we have

$$P_e \sim \frac{1}{m_e}\left(\frac{\rho}{\mu_e m_p}\right)^{4/3} \hbar c$$

$$\boxed{P_e \sim \frac{\hbar c}{(\mu_e m_p)^{4/3} m_e} \rho^{4/3}}$$

Relativistic electrons

Now we have $\gamma = 4/3 \Rightarrow$ Unstable!

To be more precise, we'll have to do some quantum/statistical mechanics.

First, we define a chemical potential of a particle,

$$\mu_i = \left(\frac{\partial E}{\partial N}\right)_{S,V} \rightsquigarrow \text{Energy cost per particle at constant entropy, volume}$$

In equilibrium, a chemical/nuclear reaction

$$A+B \rightleftharpoons C+D$$

has

$$\mu_A + \mu_B = \mu_C + \mu_D$$

In QM, particles have a distribution function in coordinate-momentum space

$$\frac{dN}{d^3x\,d^3p} = \frac{1}{h^3} \underbrace{\sum_j}_{\text{sum over possible states}} \frac{\overbrace{g_j}^{\text{degeneracy of state } j}}{\exp\left(\frac{-\mu + \varepsilon_j + \varepsilon(p)}{k_B T}\right) \pm 1} \quad \begin{matrix} + \text{ Fermi-Dirac} \\ - \text{ Bose-Einstein} \end{matrix}$$

Example: Electron in ground state in H-atom has $\varepsilon_j = -13.6\,eV$, $g_j = 2$ because there are two spin states.

The number density of particles is

$$n = \int \frac{dN}{d^3x\,d^3p}\, d^3p = \int \frac{dN}{d^3x\,d^3p}\, 4\pi p^2 dp$$

The energy of a free particle is

$$\varepsilon(p) = \left[p^2c^2 + m^2c^4\right]^{1/2}$$

In most cases the relevant energy is the particle's kinetic energy, i.e., total energy − rest mass energy

$$\mathcal{E}_j = (p^2c^2 + m^2c^4)^{1/2} - mc^2$$

If particle is non-relativistic $(p^2c^2 \ll m^2c^4)$,

$$\mathcal{E}_j \simeq \frac{1}{2}\frac{p^2}{m} \simeq \frac{1}{2}mv^2$$

Consider distribution function for electrons, which have $g_j = 2$ for $\pm 1/2$ spin. Their # density is

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp\left(\frac{-\mu_e + m_ec^2 + \mathcal{E}(p)}{k_BT}\right) + 1}$$

Note that denominator is larger than one, so that

$$\frac{dn}{dp} \leq \frac{8\pi p^2}{h^3} \quad \text{(Pauli exclusion principle)}$$

Even if $T=0$, electrons cannot have zero momentum. Note also this can be written

$$\frac{h^3}{8\pi} \leq \frac{p^2\,dp}{dn} \Rightarrow \hbar \lesssim \Delta p \Delta x$$

Can write

$$n = \frac{8\pi}{h^3}\int_0^\infty dp\, p^2\, F(\mathcal{E})$$

where

$$F(\mathcal{E}) = \left[e^{[\mathcal{E} - (\mu_e - m_ec^2)]/k_BT} + 1\right]^{-1} \quad \text{Fermi-Dirac Distribution}$$

In equilibrium $e^+ + e^- \rightarrow 2\gamma$ implies

$$\mu_{e^-} + \mu_{e^+} = 0$$

Let's define $\mu_e = \mu_{e^-} - m_ec^2$, so $\mu_{e^+} = -\mu_e - 2m_ec^2$

Then

$$n_{e^-} = \frac{8\pi}{3} \int \frac{p^2 dp}{e^{(\varepsilon - \mu_e)/k_B T} + 1}$$

$$n_{e^+} = \frac{8\pi}{3} \int \frac{p^2 dp}{e^{(\varepsilon + \mu_e + 2 m_e c^2)/k_B T} + 1}$$

Typically positrons have much lower density because $k_B T \ll 2 m_e c^2$ in most cases. Recall $\varepsilon(p)$ is kinetic energy

$$\varepsilon = (p^2 c^2 + m_e^2 c^4)^{1/2} - m_e c^2$$

$$\simeq \begin{cases} p^2/2m_e & \text{non-relativistic}, \quad p \ll m_e c \\ pc & \text{relativistic}, \quad p \gg m_e c \end{cases}$$

The pressure of the electrons is

$$P = \frac{1}{3} \int v\, p \frac{dN}{d^3x\, d^3p} d^3p$$

$$\Rightarrow P = \frac{8\pi}{3h^3} \int_0^\infty \frac{v p^3}{e^{(\varepsilon - \mu_e)/k_B T} + 1} dp$$

and the energy density is

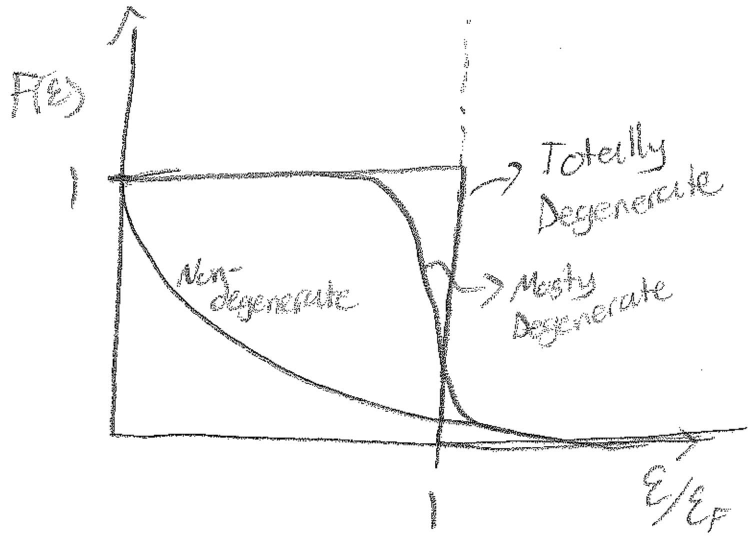
$$U = \int \varepsilon(p) \frac{dN}{d^3x\, d^3p} d^3p$$

$$\Rightarrow U = \frac{8\pi}{h^3} \int_0^\infty \frac{\varepsilon(p)\, p^2}{e^{(\varepsilon - \mu_e)/k_B T} + 1}$$

A cold degenerate plasma is defined as one where $k_B T \ll \mu$, i.e., the limit where $T \to 0$. Then

$$F(\varepsilon) = 1 \quad \text{for} \quad \varepsilon < \varepsilon_F$$
$$F(\varepsilon) = 0 \quad \text{for} \quad \varepsilon > \varepsilon_F$$

$\varepsilon_F = \mu$ is the Fermi energy



All low energy states filled for a degenerate gas.

There is corresponding Fermi momentum p_F,

$$\varepsilon_F = (p_F^2 c^2 + m_e^2 c^4)^{1/2} - m_e c^2$$

For a fully degenerate plasma, the integral over momentum states simplifies:

$$n_e = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, dp = \frac{8\pi}{3h^3} p_F^3$$

So the Fermi momentum is

$$\boxed{p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}}$$

Degeneracy effects become important when thermal energy (ideal gas) of electrons is smaller than Fermi energy, which for non-relativistic electrons happens when

$$k_B T < \frac{p_F^2}{2m_e} \Rightarrow k_B T < \frac{1}{2m_e}\left(\frac{3h^3 n_e}{8\pi}\right)^{2/3}$$

This typically happens at high densities where n_e is large

Also note ions are almost never degenerate, because their larger mass requires much larger densities for degeneracy.

Degeneracy pressure in limit $T \to 0$:

$$P_e \simeq \frac{8\pi}{3h^3} \int_0^{p_F} v\, p^3\, dp$$

Velocity is defined

$$v = \frac{\partial \mathcal{E}}{\partial p} = \frac{p}{m}\left(1 + \frac{p^2}{m^2c^2}\right)^{-1/2}$$

In non-relativistic limit $p \ll mc$,

$$v_e \simeq \frac{p}{m_e}$$

$$\Rightarrow P_e \simeq \frac{8\pi}{3h^3 m_e} \int_0^{p_F} p^4\, dp$$

$$\Rightarrow P_e \simeq \frac{8\pi p_F^5}{15 h^3 m_e}$$

$$\Rightarrow \boxed{P_e \simeq \left(\frac{3}{8\pi}\right)^{2/3} \frac{1}{5} \frac{h^2}{m_e} n_e^{5/3}}$$

Non-relativistic degeneracy EOS

And the internal energy density is

$$U \simeq \frac{8\pi}{h^3} \int_0^{p_F} \mathcal{E}(p)\, p^2\, dp \simeq \frac{4\pi}{h^3 m_e} \int p^4\, dp$$

$$\Rightarrow U \simeq \frac{4\pi p_F^5}{5h^3 m_e} \simeq \frac{3}{2} P_e$$

Do relativistic limit in homework!

White Dwarfs

Most white dwarfs can be approximated as being supported entirely by non-relativistic degeneracy pressure. From this fact, we can calculate most of their properties.

The internal energy/pressure is in the degenerate electrons, which have

$$\mathcal{E}_e \sim \mathcal{E}_F \sim \frac{\hbar^2 n_e^{2/3}}{m_e}$$

The mass of the star is

$$M = N_e \mu_e m_p \Rightarrow N_e = \frac{M}{\mu_e m_p}$$

Its total internal energy is

$$E = N_e \mathcal{E}_e$$
$$= \frac{M}{\mu_e m_p} \frac{\hbar^2 n_e^{2/3}}{m_e}$$

Using

$$n_e \sim \frac{N_e}{R^3} \sim \frac{M}{\mu_e m_p R^3}$$

$$E \sim \frac{\hbar^2}{m_e (\mu_e m_p)^{5/3}} \frac{M^{5/3}}{R^2}$$

From the Virial theorem,

$$E \sim E_g \sim \frac{GM^2}{R}$$

$$\Rightarrow \frac{GM^2}{R} \sim \frac{\hbar^2}{m_e (\mu_e m_p)^{5/3}} \frac{M^{5/3}}{R^2}$$

$$\Rightarrow \boxed{R \sim \frac{\hbar^2}{G m_e (\mu_e m_p)^{5/3}} M^{-1/3}}$$

More massive white dwarfs are smaller!

This evaluates to, for $\mu_e = 2$,

$$R \approx 10^{-2} R_\odot \left(\frac{M}{M_\odot}\right)^{-1/3}$$

White dwarfs are small!

As the mass is increased, so does the electron density

$$n_e \propto \frac{M}{R^3} \propto M^2$$

and so does the Fermi energy and momentum. Electrons begin to become relativistic for WDs with $M \gtrsim 1 M_\odot$. For a relativistic population of electrons

$$\mathcal{E}_F \sim p_F c \sim \hbar n_e^{1/3} c$$

In this case the internal energy is

$$E \sim \frac{M}{\mu m_p} \hbar n_e^{1/3} c$$

$$\sim \frac{\hbar c}{(\mu_e m_p)^{4/3}} \frac{M^{4/3}}{R}$$

Equating to the gravitational energy,

$$\Rightarrow \frac{GM^2}{R} \sim \frac{\hbar c M^{4/3}}{(\mu_e m_p)^{4/3} R}$$

$$\Rightarrow \boxed{M \sim \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{(\mu_e m_p)^2} \sim M_{Ch}}$$

Chandrasekhar mass

For $\mu_e = 2$, $M_{Ch} \simeq 1.4 M_\odot$

White dwarfs cannot be more massive than this!

For neutron stars, the derivation proceeds almost the same, but with $\mu_e m_p \rightarrow \mu_N m_N \simeq m_N$. So we get $M_{max} \sim 5.6 M_\odot$ for neutron stars. Detailed calculations suggest $M_{max} \lesssim 3 M_\odot$ for NSs.

Crystallization

In white dwarfs, the electrons are degenerate, but the ions are not. However, they are closely packed, and "cold", relatively speaking, because there is no internal energy source.

We must consider Coulomb interactions between ions. The Coulomb potential energy between two ions is

$$E_c = \frac{Z^2 e^2}{\underbrace{a}_{\text{separation}}}$$

If the Coulomb energy is larger than the thermal energy of the ions, electrostatic forces become important. Define

$$\Gamma_c = \frac{Z^2 e^2}{a \underbrace{k_B T}_{\text{kinetic energy}}}$$

For $\Gamma_c \gtrsim 1$ we expect ideal gas law to break down.

Using $a = \left(\frac{3}{4\pi n_{ion}}\right)^{1/3}$, $n_{ion} \simeq \frac{\rho}{2Z m_p} \Rightarrow a \simeq \left(\frac{3Z m_p}{2\pi \rho}\right)^{1/3}$

$$\Rightarrow \Gamma_c \simeq \left(\frac{2\pi}{3m_p}\right)^{1/3} \frac{Z^{5/3} e^2}{k_B T} \rho^{1/3}$$

$$\simeq \left(\frac{2\pi Z^5}{3m_p}\right)^{1/3} \frac{\alpha \hbar c}{k_B T} \rho^{1/3}$$

For WDs made of C/O, $Z \sim 7$

$$\Gamma_c \sim 44 \left(\frac{T}{10^7 K}\right)^{-1} \left(\frac{\rho}{10^6 g/cm^3}\right)^{1/3}$$

Detailed modeling shows that when $\Gamma_c \gtrsim 100$, the Coulomb interaction between ions causes them to form a lattice, i.e., the white dwarf starts to crystallize!

So, as white dwarfs cool, they eventually crystallize their cores (though electrons remain degenerate gas). Because WD interiors are nearly isothermal, crystallization starts at the center where ρ is highest, and the crystalline core grows outward.