Ay 123 Lecture V Homology and the Main Sequence

Goal: Develop scaling relations between stellar properties, e.g., how R, T_{eff}, L scale with mass.

To do this, we'll have to make some assumptions about the star:

- chemical homogeneity
- radiative or convective energy transport
- equation of state, gas vs. radiation pressure
- opacity law
- nuclear energy generation

Equations of stellar structure:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad 5.1$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \qquad 5.2$$

$$\frac{dL}{dm} = \epsilon \qquad 5.3$$

$$\frac{dT}{dm} = -\frac{3\kappa L}{64\pi^2 ac r^4 T^3} \qquad 5.4$$

5.4 assumes radiative energy transport. You'll consider convective energy transport in the homework.

In general, solving these equations requires numerical computation.

Here we will derive scaling relations, i.e., homology relations,

A homology relation is one that scales with the mass of the star, e.g.,

$$R = \bar{R}(M_*)\left(\frac{M}{M_*}\right)^{a}$$

where M_* is a reference mass, and a is a constant to be determined. Let's assume

$$R = \bar{R}(M_*)\left(\frac{M}{M_*}\right)^{a}$$

$$\rho = \bar{\rho}(M_*)\left(\frac{M}{M_*}\right)^{b}$$

$$L = \bar{L}(M_*)\left(\frac{M}{M_*}\right)^{c}$$

$$T = \bar{T}(M_*)\left(\frac{M}{M_*}\right)^{d}$$

$$P = \bar{P}(M_*)\left(\frac{M}{M_*}\right)^{e}$$

We solve for constants a-e by invoking stellar structure equations, Equation 5.1 becomes

$$5.1 \rightarrow \quad M_*^{e-1}\frac{d\bar{P}}{dm} = -G M_*^{1-4a}\frac{m}{4\pi \bar{r}^4}$$

This is true for any mass, so this requires powers of M_* on each side to be equal.

So

$$e-1 = 1-4a$$

$$\Rightarrow \boxed{4a+e=2}$$

Next, equation 5.2:

$$5.2 \rightarrow M_*^{a-1} \frac{d\bar{r}}{dm} = \frac{1}{4\pi M_*^{2a+b} \bar{r}^2 \bar{\rho}}$$

$$\Rightarrow a-1 = -2a-b$$

$$\Rightarrow \boxed{3a+b=1}$$

Next, we introduce scaling relations for ϵ and K.

$$\epsilon = \epsilon_0 \rho T^{\eta}$$

$$K = K_0 \rho^{\lambda-1} T^{3-\nu}$$

λ, ν, η are constants that depend on form of nuclear energy generation and opacity law.

Equation 5.3:

$$5.3 \longrightarrow M_*^{c-1} \frac{d\bar{L}}{dm} = \epsilon_0 M_*^{b+\eta d} \bar{\rho} \bar{T}^{\eta}$$

$$\Rightarrow c-1 = b+\eta d$$

$$\Rightarrow \boxed{c = 1+b+\eta d}$$

Equation 5.4:

$$M_*^{d-1} \frac{d\bar{T}}{dm} = -3K_0 M_*^{c+(\lambda-1)b-4a-\nu d} \frac{\bar{\rho}^{\lambda-1} \bar{L}}{64\pi^2 ac \bar{r}^4 \bar{T}^{\nu}}$$

$$\Rightarrow d-1 = c+(\lambda-1)b-4a-\nu d$$

$$\Rightarrow \boxed{4a+(\nu+1)d = (\lambda-1)b+c+1}$$

Last equation provided by equation of state:

For ideal gas,

$$P = \frac{\rho k_B T}{\mu m_p}$$

$$\Rightarrow M_*^{e}\bar{P} = M_*^{b+d}\frac{\bar{\rho} k_B \bar{T}}{\mu m_p}$$

$$\Rightarrow \boxed{e = b+d}$$ For gas pressure

For radiation pressure,

$$P = \frac{1}{3}aT^4$$

$$\Rightarrow M_*^{e}\bar{P} = \frac{1}{3}a M_*^{4d}\bar{T}^4$$

$$\Rightarrow \boxed{e = 4d}$$ For radiation pressure

Given a n, λ, ν, we have 5 equations for 5 unknowns, and have a linear algebra problem for a–e. Note this will not give $\bar{\rho}, \bar{T}$, etc, but only how ρ, T, etc, scale with (M/M_*)

i. Mass-Luminosity relation:

$$L = \bar{L}\left(\frac{m}{M}=1\right)\left(\frac{M}{M_*}\right)^{c}$$

$\hookrightarrow$ Same for all stars in homology class

$$\Rightarrow L \propto M^{c}$$

ii. Luminosity – T_{eff} Relation (HR Diagram)

$$R = R_\star \left(\frac{M}{M_\star}\right)^a$$

$$L = 4\pi\sigma R^2 T_{eff}^4$$

$$\Rightarrow T_{eff} = \left[\frac{L}{4\pi\sigma R^2}\right]^{1/4} = T_{eff,\star} \left(\frac{M}{M_\star}\right)^{(c-2a)/4} \Rightarrow M \propto T_{eff}^{\frac{4}{c-2a}}$$

and $$L \propto \left(\frac{M}{M_\star}\right)^c \propto T_{eff}^{\frac{4c}{c-2a}} \propto T_{eff}^f \qquad f = \frac{4c}{c-2a}$$

iii. Mass-radius

$$R = R_\star \left(\frac{M}{M_\star}\right)^a \Rightarrow R \propto M^a$$

We still haven't specified λ, ν, n.

For intermediate and low-mass stars, dominant form of opacity is free-free and bound-free absorption.

Both are Kramer's opacity:

$$\kappa = \kappa_0 \rho T^{-3.5}$$

So for these stars $\lambda = 2$, $\nu = 6.5$

For high-mass stars, electron scattering dominates

$$\kappa = \kappa_0 \quad \text{(independent of } T, \rho)$$

$$\Rightarrow \lambda = 1, \quad \nu = 3$$

Nuclear energy generation

$$\epsilon = \epsilon_0 \rho T^4, \quad n=4$$ for low-mass stars with pp-chain

High-mass stars
Energy generation by CNO cycle

$$\epsilon = \epsilon_0 \rho T^{17} \Rightarrow n=17$$

Now we put each of our equations into a line of a matrix:

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & n & 0 \\ 4 & 1-\lambda & -1 & \nu+1 & 0 \\ 0 & \underbrace{1 \text{ or } 0} & 0 & \underbrace{1 \text{ or } 4}_{\text{Gas or radiation pressure}} & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Solve matrix equation for a–e:

	ν	λ	n	P	a	b	c	d	e	f
Low-mass	6.5	2	4	gas	1/13	10/13	71/13	12/13	22/13	284/69
Med-mass	6.5	2	17	gas	0.69	-1.07	5.15	0.31	-0.77	5.47
High-mass	3	1	17	gas	4/5	-7/5	3	1/5	-6/5	60/7
Very high-mass	3	1	17	rad	19/40	-17/40	1	1/40	1/10	80

The radiation pressure-supported stars probably don't really exist: they are unstable

How do these exponents compare with observations?

Observed

i. $L \propto M^c \propto M^{3-5.5}$ — $L \propto M^{3.9}$

ii. $L \propto T_{eff}^f \propto T_{eff}^{4-9}$ — Higher mass stars have steeper L T relation

iii. $R \propto M^a \propto M^{0.1-0.8}$ — $R \propto M^{0.75}$ shallower for lowest mass stars

Relations are not perfect, but help us understand basic scaling

Lifetime on main sequence. We expect

$$\tau \propto \frac{M}{L} \propto M^{1-c} \propto M^{-2 \text{ to } -4.5}$$

Lower-mass stars have much longer lifetimes!

For a population of stars with the same age, (eg, a star cluster), there is a mass M_{turn} at which stars above this mass have left the main sequence, while stars below are on main sequence. This is the MS turn-off mass, If M_{turn} is measured, we can calculate age of all stars in the cluster.

Minimum mass for H-burning:

$$T_{core} \propto M^d \tilde{T}_{core} \propto M^{0.9} \quad \text{for low-mass stars}$$

For the Sun, $T_{core} \simeq 1.5 \times 10^7$ K

Minimum T_{core} for H-burning is $T_{min} \simeq 4 \times 10^6$ K

So $$\left(\frac{M_{min}}{M_\odot}\right)^{0.9} = \frac{T_{min}}{T_{\odot,core}}$$

$$\Rightarrow M_{min} \simeq 0.2\, M_\odot \quad \text{(Detailed models give} \sim 0.1\, M_\odot)$$

Homology relations could be expanded to include dependence on composition, but usually dependence is weak.

Stars not on main sequence must violate assumptions:

Giant stars: not chemically homogeneous

White dwarfs: no nuclear burning

Sub-dwarfs (helium stars): Different composition than MS