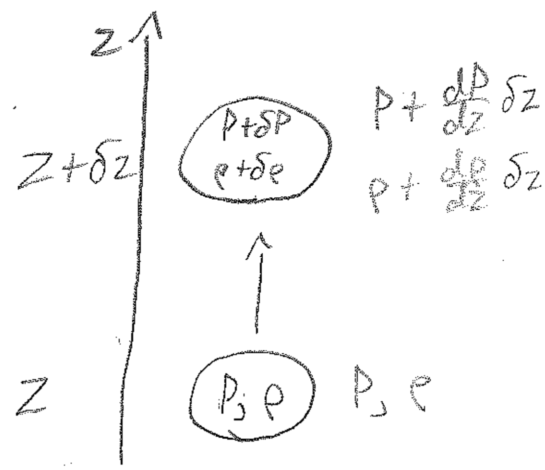
Ay 123 Lecture III Part b

Convection

Conduction/radiative diffusion happens when $\frac{dT}{dr} < 0$

Convection occurs if $dT/dr <$ critical value

Consider parcel of gas that moves within star, such that it maintains pressure balance with surrounding fluid. Because $\tau_{therm} \gg \tau_{dyn}$, we'll consider the limit where the parcel does not exchange heat with its surroundings, such that its motion is adiabatic.



For adiabatic expansion/compression

$$PV^{\gamma} = \text{constant} \propto S \hookrightarrow \text{entropy}$$

If composition is uniform,

$$P \propto \rho^{\gamma}$$

$$\Rightarrow \frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho}$$

If parcel maintains pressure equilibrium,

$$\delta P = \frac{dP}{dz}\delta z$$

$$\Rightarrow \frac{\gamma P}{\rho}\delta\rho = \frac{dP}{dz}\delta z$$

Will parcel be lighter than its surrounding material and continue to rize, or be heavier and fall back?

Parcel will continue to rise if it is less dense, i.e.,

$$\rho + \delta\rho < \rho + \frac{d\rho}{dz}\delta z$$

$$\Rightarrow \frac{\rho}{\gamma P}\frac{dP}{dz}\delta z < \frac{d\rho}{dz}\delta z$$

$$\Rightarrow \frac{d\ln P}{d\ln z} < \gamma \frac{d\ln\rho}{d\ln z}$$ Convectively unstable

Note we can also write this as, using $C_s^2 = \frac{\gamma P}{\rho}$

$$\frac{dP}{dz} < C_s^2 \frac{d\rho}{dz}$$

Or, using $dP/dz = -\rho g$

$$\Rightarrow -\rho g < C_s^2 \frac{d\rho}{dz}$$

$$\Rightarrow \frac{d\ln\rho}{d\ln z} < \frac{gz}{C_s^2}$$ Unstable

The acceleration due to buoancy forces is

$$a = -\frac{\delta\rho - \frac{d\rho}{dz}\delta z}{\rho} g$$

$$= -\left(\frac{1}{\gamma P}\frac{dP}{dz} - \frac{1}{\rho}\frac{d\rho}{dz}\right) g\,\delta z$$

$$= -N^2 \delta z$$

where N is the Brunt-Väisälä frequency,

$$N^2 = \left(\frac{1}{\gamma}\frac{d\ln P}{dr} - \frac{d\ln \rho}{dr}\right) g$$

$N^2 > 0$ Stably stratified

$N^2 < 0$ Convectively Unstable

or using hydrostatic equilibrium,

$$N^2 = \frac{-g^2}{c_s^2} - g\frac{d\ln \rho}{dr}$$

Recall that an adiabatic perturbation has constant entropy, i.e.,

$$\delta S = \delta\left(P/\rho^{\gamma}\right) = 0$$

for constant composition. Then we can write

$$S \propto \frac{P}{\rho^{\gamma}}$$

$$\Rightarrow \frac{1}{S}\frac{dS}{dr} = \frac{1}{P}\frac{dP}{dr} - \gamma\frac{1}{\rho}\frac{d\rho}{dr}$$

$$\Rightarrow \frac{1}{\gamma}\frac{d\ln S}{dr} = \frac{1}{\gamma}\frac{d\ln P}{dr} - \frac{d\ln \rho}{dr}$$

And so

$$N^2 = \frac{g}{\gamma}\frac{d\ln S}{dr}$$

So a positive entropy gradient is stable, i.e., warm material on top of cold material.

A negative entropy gradient is unstable, i.e., cold material on top of warm material

Note that for an ideal gas of constant composition

$$P \propto \rho T$$

$$\Rightarrow \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\Rightarrow \frac{d\ln\rho}{dr} = \frac{d\ln P}{dr} - \frac{d\ln T}{dr}$$

So we can write

$$N^2 = \left[\left(\frac{1}{\gamma} - 1\right)\frac{d\ln P}{dr} + \frac{d\ln T}{dr}\right]g$$

$$= g\frac{d\ln P}{dr}\left[\frac{1-\gamma}{\gamma} + \frac{d\ln T}{d\ln P}\right] \qquad \frac{d\ln T}{d\ln P} \equiv \nabla$$

Note also for ideal gas

$$S \propto \frac{P}{\rho^\gamma} \propto P^{1-\gamma} T^\gamma$$

$$\Rightarrow \frac{dS}{S} = (1-\gamma)\frac{dP}{P} + \gamma\frac{dT}{T} = 0 \text{ at constant entropy}$$

$$\Rightarrow \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma - 1}{\gamma} \qquad \left(\frac{\partial \ln T}{\partial \ln P}\right)_S \equiv \nabla_{ad} = \frac{2}{5} \text{ for ideal gas}$$

So we have

$$N^2 = \frac{\rho g^2}{P}\left(\nabla_{ad} - \nabla\right) \quad \text{for ideal gas}$$

Steep temperature gradient leads to convective instability

- Occurs near centers of many stars because nuclear energy generation is peaked at center, creating large ∇

- also occurs where opacity is large, because from radiative diffusion equation,

$$\nabla \propto K$$

This frequently leads to convection in ionization zones where bound-free opacity becomes large.

Convective Energy Transport

- We have seen when convection occurs, but not how much energy it transports
- Convection is a linear instability such that a fluid parcel accelerates as

$$\delta \ddot{z} = -N^2 \delta z$$

What is non-linear saturation of instability? We don't really know!

Mixing Length Theory (MLT)

- attempt to parameterize saturated state of convective instability
- MLT assumes fluid parcels are only able to remain coherent over a "mixing length" parameterized as

$$\underbrace{\ell}_{\text{Mixing Length}} = \alpha H \rightarrow \text{scale height}$$

α → parameter, $\alpha \approx 2$ from calibrated observations

- Larger α means parcels move farther, such that energy transport is more efficient

The excess thermal energy carried by a rising blob is

$$\epsilon = C_p \rho \Delta T \quad erg/cm^3$$

↳ Temp excess over background

So the convective flux is

$$F_{con} = C_p \rho \Delta T V_{con}$$

→ velocity of parcel, "convective speed"

As parcel rises, its velocity and temperature excess grow:

$$\frac{d}{dt} V_{con} = \sqrt{-N^2} V_{con} \simeq \frac{g \Delta T}{T}$$

$$\Rightarrow V_{con} \sim \sqrt{-N^2}\, l \sim \sqrt{-N^2}\, \alpha H$$

And similarly $$\Delta T \sim T \frac{\sqrt{-N^2}}{g} V_{con} \sim \frac{-N^2 \alpha H}{g}$$

So

$$F_{con} \sim \frac{C_p T \rho |N|^3 (\alpha H)^2}{g}$$

or

$$F_{con} \sim \frac{C_p T \rho}{g \alpha H} V_{con}^3$$

Recall $H = P/\rho g$

$$\Rightarrow F_{con} \sim \frac{C_p T \rho}{\alpha P} \rho V_{con}^3$$

Using $C_p \sim \frac{k_B}{\mu m_p}$, for ideal gas

$$F_{con} \sim \frac{\rho V_{con}^3}{\alpha}$$

Let's assume that convection carries all the luminosity in some part of a star. Then

$$L = L_{con} = 4\pi r^2 F_{con}$$
$$= \frac{4\pi r^2 C_P T \rho |N|^3 \alpha^2 H^2}{g}$$

Using $H = P/\rho g$, $|N| = \frac{\rho^{1/2} g}{P^{1/2}} (\nabla_{ad} - \nabla)^{1/2}$

$$L = 4\pi \alpha^2 (\nabla_{ad} - \nabla)^{3/2} C_P T \rho \frac{\rho^{3/2} g^3}{P^{3/2}} \frac{P^2}{\rho^2 g^2} \frac{r^2}{g}$$
$$= 4\pi \alpha^2 (\nabla_{ad} - \nabla)^{3/2} C_P T \rho \frac{P^{1/2}}{\rho^{1/2}} r^2$$

Using $c_s^2 = \gamma P/\rho$, $C_P = \frac{5}{2} \frac{k_B}{\mu m_p}$

$$L = 4\pi \alpha^2 \gamma^{-1/2} \frac{5}{2} P c_s r^2 (\nabla_{ad} - \nabla)^{3/2}$$

Recall that to order of magnitude, deep inside a star,

$$P \sim \frac{GM^2}{r^4}, \quad c_s \sim \sqrt{\frac{GM}{r}}$$

Then

$$L \sim \frac{G^{3/2} M^{5/2}}{R^{5/2}} (\nabla_{ad} - \nabla)^{3/2}$$
$$\sim \frac{GM^2}{R} \sqrt{\frac{GM}{R^3}} (\nabla_{ad} - \nabla)^{3/2}$$
$$\sim \frac{W}{\tau_{dyn}} (\nabla_{ad} - \nabla)^{3/2}$$

Result:
$\boxed{\nabla \sim \nabla_{ad} \text{ in convective regions}}$

But recall $L \sim W/\tau_{therm}$

$$\Rightarrow (\nabla_{ad} - \nabla) \sim \left(\tau_{dyn}/\tau_{therm}\right)^{2/3}$$

In most of stellar interior, $\tau_{dyn} \ll \tau_{therm}$, so $(\nabla_{ad} - \nabla) \ll 1$