Ay101 Set 4 solutions

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1. **Protostar**

   (a) **Find the average density and central temperature (as a function of mass) of an accreting protostar whose initial radius is given by the expression**

$$\frac{R}{R_\odot} = \frac{43.2}{1 - 0.2X}\frac{M}{M_\odot}$$

if its structure is approximated by a $n = 1.5$ polytrope with hydrogen mass fraction $X = 0.7$ and helium fraction $Y = 0.3$.
The average density is given by $\bar{\rho} = \frac{3}{4\pi}\frac{M}{R^3}$. We know that the radius is given by $R = \frac{43.2}{1-0.2X}\frac{M}{M_\odot}R_\odot$. Substituting $X = 0.7$ and the solar values, we find that $R = (1.76 \times 10^{-21}\ \text{cm/g})M$. The average density is then given by

$$\bar{\rho} = \frac{3}{4\pi}\frac{M}{(1.76 \times 10^{-21}\ \text{cm/g})^3 M^3}$$

$$\boxed{\bar{\rho} = (1.1 \times 10^{-5}\ \text{g/cm}^3)\left(\frac{M}{M_\odot}\right)^{-2}}$$

The central temperature is given by $T_c = C\frac{\mu M}{R}$ where $C = 4.347 \times 10^{-16}$ (HKT Eq. 7.41). One of the easiest ways to compute mean molecular weight is using $\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$, which yields $\mu = 0.62$. Plugging in numbers (fortunately M appears in both the numerator and denominator and cancels out), we find $\boxed{T_c = 1.52 \times 10^5\ \text{K}}$.

   (b) **Suppose the protostar contracts to smaller radius (at constant mass) but maintains a polytropic structure until its collapse is halted when the central temperature reaches $T_{\textbf{crit}}$ required for hydrogen burning. Show that the greater the mass of the star, the smaller the density at the point where $T_{\textbf{crit}}$ is reached:**

$$\rho_{\textbf{crit}} = 1.52\frac{1}{M^2}\left(\frac{k_B T_{\textbf{crit}}}{\mu m_H G}\right)^3$$

There are several ways to do this, but I recommend starting with the polytrope equations given in HKT (Eqs. 7.37-7.42). Perhaps the most straightforward way to do this is to consider the equation for central temperature (HKT Eq. 7.41, noting that Avogadro's number N_A is roughly the reciprocal of m_p):

$$T_\text{crit} = \frac{1}{(n+1)(-\xi\theta')_{\xi_1}}\frac{G\mu m_p}{k_B}\frac{M}{R}$$

Plug in the expression for R as a function of $\bar{\rho} = \rho_\text{crit}$, then solve for ρ_crit:

$$T_\text{crit} = \frac{1}{(n+1)(-\xi\theta')_{\xi_1}}\frac{G\mu m_p}{k_B}M\left(\frac{3M}{4\pi\rho_\text{crit}}\right)$$

$$\rho_\text{crit} = \frac{3}{4\pi M^2}\left(\frac{k_B T_\text{crit}(n+1)(-\xi\theta')_{\xi_1}}{G\mu m_p}\right)^3$$

Plugging in numbers (from HKT Table 7.1), we find that $\boxed{\rho_{\text{crit}} = 1.52\frac{1}{M^2}\left(\frac{k_B T_{\text{crit}}}{\mu m_p G}\right)^3}$, as expected.

(c) **Noting the criterion for electron degeneracy, estimate the critical mass below which collapse is halted by electron degeneracy, not by hydrogen burning. After dropping factors of order unity, show that this mass is related to the Chandrasekhar limit, M_{Ch}, by the approximate relation**

$$\frac{M_{\textbf{crit}}}{M_{Ch}} \sim \left(\frac{\mu_e}{\mu}\right)^{3/2} \left(\frac{k_B T_{\textbf{crit}}}{m_e c^2}\right)^{3/4}$$

Evaluate this mass for $T_{\textbf{crit}} = 5 \times 10^6$ K and $M_{Ch} = 1.4\ M_\odot$.

The criterion for electron degeneracy is that (degeneracy energy) > (thermal energy):

$$\frac{p_F^2}{2m_e} > k_B T$$

$$\frac{1}{2m_e}\left(\frac{3h^3 n_e}{8\pi}\right)^{2/3} > k_B T$$

Note that electron number density $n_e = \frac{\rho_{crit}}{\mu_e m_p}$, where ρ_{crit} is the critical density from the previous portion. Plugging these in and solving for M_{crit}, we find:

$$M_{\text{crit}} = \left(\frac{k_B T_{\text{crit}}}{2m_e}\right)^{3/4} \left(\frac{(1.52)3}{8\pi\mu_e}\right)^{1/2} \left(\frac{h}{\mu G}\right)^{3/2} m_p^{-2}$$

Dividing this by the Chandrasekhar mass $M_{Ch} = \left(\frac{hc}{2\pi G}\right)^{3/2} \frac{1}{(\mu_e m_p)}^2$, we find

$$\frac{M_{\text{crit}}}{M_{Ch}} = \left(\frac{1}{2}\right)^{3/4} \left(\frac{(1.52)3}{8\pi}\right)^{1/2} (2\pi)^{-3/2} \frac{\mu_e^{3/2}\mu^{-3/2}(k_B T_{\text{crit}})^{3/4}}{(m_e c^2)^{3/4}}$$

Fortunately, all those numbers at the beginning are of order unity, so we can drop them:

$$\boxed{\frac{M_{\text{crit}}}{M_{Ch}} \sim \left(\frac{\mu_e}{\mu}\right)^{3/2} \left(\frac{k_B T_{\text{crit}}}{m_e c^2}\right)^{3/4}} \tag{1}$$

For our object, $\mu = 0.62$ and μ_e is the number of baryons per electron. This is given by

$$\mu_e = \frac{1}{\sum \frac{\#\text{ electrons}}{\text{baryon}} \times (\text{mass fraction})} = \frac{1}{X + \frac{1}{2}Y + \frac{1}{2}Z} = 1.18$$

Plugging in this value of μ along with the given values of T_{crit} and M_{Ch} into Equation (2), we find that $\boxed{M_{\text{crit}} \sim 0.018\ M_\odot}$, which is approximately 20 Jupiter masses. Our object is a brown dwarf!

2. **Supernova Shock Revival from Neutrino Heating**

(a) **Estimate the energy that is required to photodissociate 0.8 $M_\odot$ of Fe into alpha particles and neutrons. Compare this energy to the bounce shock energy and comment on the fate of the shock.**

First find what particles ^{56}Fe dissociates into: $^{56}\text{Fe} = 26\text{p} + 30\text{n} = 13\alpha + 4\text{n}$

Then find the energy needed to dissociate a single ^{56}Fe nucleus:

$$\begin{aligned} Q &= (13m_\alpha) + 4m_n - m(^{56}\text{Fe})c^2 \\ &= (56 - 55.85)m_p c^2 \\ &= 2.25 \times 10^{-4} \text{ erg/nucleus} \end{aligned}$$

Now calculate the total energy needed to photodissociate 0.8 $M_\odot$ of ^{56}Fe:

$$
\begin{aligned}
E_{\rm phot} &= Q(0.8\ M_\odot) \\
&= (2.25\times10^{-4}\ {\rm erg}/\cancel{\rm nucleus})(0.8\ \cancel{M_\odot})(2\times10^{33}\ \cancel{\rm g}/\cancel{M_\odot})(55.85\times1.67\times10^{-24}\ \cancel{\rm g}/\cancel{\rm nucleus})^{-1}
\end{aligned}
$$

This yields $\boxed{E_{\rm phot} = 3.9\times10^{51}\ {\rm erg}}$. The photodissociation energy is larger than the energy of the bounce $E_{\rm bounce} = 10^{51}$ erg, so the shock will not survive with its initial energy.

(b) **In the proto-neutron star (with an initial radius 2×10^6 cm), the mean free path of neutrinos is $l_\nu = 30$ cm. Estimate the diffusion time for neutrinos to escape from the proto-neutron star and hence estimate the neutrino luminosity during the initial neutron-star cooling phase.**

The diffusion time is given by $t_{\rm diff} = \frac{R^2}{lc}$, where $R = 2\times10^6$ cm is the radius and $l = 30$ cm is the mean free path. Plugging in numbers yields $\boxed{t_{\rm diff} = 4.44\ {\rm s}}$.

The neutrino luminosity L_ν is generated by the neutrinos radiating the proto-neutron star's gravitational binding energy $E_{\rm bind} \sim \frac{GM_{\rm core}^2}{R}$. Then $L_\nu \sim E_{\rm bind}/t_{\rm diff}$. Plugging in numbers yields $\boxed{L_\nu = 1.5\times10^{52}\ {\rm erg}}$.

(c) **Assuming that 10% of the neutrino luminosity is absorbed by the infalling outer core, estimate how long it takes to absorb enough neutrino energy to reverse the infall of the 0.8 $M_\odot$ outer core and drive a successful supernova explosion with a typical explosion energy of 10^{51} erg. Assume the outer core has initial energy per unit mass $\epsilon = -GM_{Fe}/R_{Fe}$. Compare this time to the dynamical (free-fall) timescale of the proto-neutron star.**

Assuming 10% of the neutrino luminosity is absorbed by the core, the total energy absorbed is $0.1L_\nu t$:

$$
0.1L_\nu t = E_{\rm infall} + E_{\rm SN}. \tag{2}
$$

Here, $E_{\rm SN} = 10^{51}$ erg is the energy of the supernova and $E_{\rm infall}$ is the infall energy of the outer core. The total infall energy is given by

$$
\begin{aligned}
E_{\rm infall} &= \epsilon M_{\rm outer\ core} \\
&= \frac{GM_{\rm Fe}}{R_{\rm Fe}}(0.8\ M_\odot) \\
&= 1.07\times10^{51}\ {\rm erg}
\end{aligned}
$$

Solving equation (1) for the time, we find

$$
\begin{aligned}
t &= \frac{E_{\rm infall}+E_{\rm SN}}{0.1L_\nu} \\
&= \frac{(1.07+1)\times10^{51}\ {\rm erg}}{0.1(1.5\times10^{52}\ {\rm erg/s})}
\end{aligned}
$$

which yields $\boxed{t = 1.4\ {\rm s}}$.

Compare this to the dynamical (free-fall) time of $\boxed{t_{\rm dyn} = \sqrt{\frac{R^3}{GM}} = 2.9\times10^{-4}\ {\rm s}}$. It takes many dynamical timescales to drive the explosion!

(d) **Once the star explodes, the duration of the observed supernova is determined by the photon diffusion time through the expanding ejecta. Provide an order of magnitude estimate of this diffusion time $t_{\rm diff}$, as a function of the ejecta mass M and the explosion kinetic energy E, assuming pure helium ejecta and electron scattering opacity. Remember that the radius of the ejecta is increasing with time as $R = vt$, where v is the ejecta velocity. The duration of the observed supernova is approximately the time at which $t = t_{\rm diff}$. Evaluate $t_{\rm diff}$ for an ejecta mass of 5 $M_\odot$ and $E = 10^{51}$ erg.**

First, let's relate opacity with diffusion time. As we know (see problem 2b), $t_{\text{diff}} = \frac{R^2}{lc}$, where l is the mean free path.

We can also define the mean free path as $l = \frac{1}{\kappa\rho}$, so we now have $t_{\text{diff}} = \frac{R^2\kappa\rho}{c}$.

Then plug in $\rho = \frac{3M}{4\pi R^3}$ to find: $t_{\text{diff}} = \frac{3\kappa M}{4\pi Rc}$.

Now, since the radius of the ejecta is increasing with time ($R = vt$), we have $t_{\text{diff}} = \frac{3\kappa M}{4\pi vtc}$.

But since v is set by the kinetic energy of the explosion ($E = \frac{1}{2}Mv^2$), we know $v = \sqrt{2E/M}$: $t_{\text{diff}} = \frac{3\kappa M}{4\pi\sqrt{2E/M}tc} = \frac{3\kappa M^{3/2}}{4\pi\sqrt{2E}tc}$.

And $t \sim t_{\text{diff}}$ at the duration of the supernova, so $t_{\text{diff}}^2 = \frac{3\kappa M^{3/2}}{4\pi\sqrt{2E}c} \Rightarrow \boxed{t_{\text{diff}} = \left(\frac{3\kappa}{4\pi\sqrt{2}c}\right)^{1/2} M^{3/4}E^{-1/4}}$.

Recall that electron-scattering opacity is given by $\kappa_{\text{es}} = 0.2(1+X)$ cm^2 g^{-1}. We're considering pure helium, so $X = 0$ and $\kappa_{\text{es}} = 0.2$ cm^2 g^{-1}.

Plugging in all the numbers, we find $\boxed{t_{\text{diff}} = 69 \text{ days.}}$

3. **Stellar Spectra, Part 1**

   See attached Jupyter notebook `Ay101_ps4-Solved.ipynb` for solutions.