Problem 1 - Coordinates & Observations

This is an exercise in coordinate transformation: Given the equatorial coordinates of Sgr A*, $\alpha = 17^h 45^m 40.045 \pm 0.01 \text{ s}$, $\delta = -29^\circ 00' 27.9 \pm 0.2''$ we can use the following equations (e.g. Binney & Merrifield [BM] Eq (2.1)) to convert to galactic coordinates,

\[
\sin \delta = \sin \delta_{GP} \sin \delta + \cos \delta_{GP} \cos \delta \cos (\alpha - \alpha_{GP}) \tag{1.1}
\]
\[
\cos b \cos (l_{NCP} - l) = \cos \delta \sin (\alpha - \alpha_{GP}) \tag{1.2}
\]
\[
\cos b \cos (l_{NCP} - l) = \cos \delta_{GP} \sin \delta - \sin \delta_{GP} \cos \delta \cos (\alpha - \alpha_{GP}) \tag{1.3}
\]

Using given values: Declination of Galactic North Pole $\delta_{GP} = 27.12825^\circ$, RA (in deg) of Galactic North Pole $\alpha_{GP} = 192.85948^\circ$, Galactic longitude of Celestial North Pole $l_{NCP} = 123.932^\circ$, Sgr A* RA $\alpha = 266.41685^\circ$, Sgr A* Dec $\delta = -29.00775^\circ$

[* Actually, $l_{NCP} = 122.932^\circ$, so the value given in the question is off by a whole degree, but for our pedagogical purposes it makes no difference as long as your answer is consistent with your assumed value.]*

Plugging in these values, equations (1.1), (1.2) and (1.3) give

\[
\sin b = -0.000805346 \quad \text{or} \quad b = -0.04614^\circ
\]
\[
\sin (123.932 - l) = 0.838788
\]
\[
\cos (123.932 - l) = -0.544458
\]

From the signs of $\sin$ and $\cos$ we infer $(123.932 - l)$ lies in the second quadrant and we get $l = 0.94437^\circ$ \[ or $l = -0.0056^\circ$ for $l_{NCP} = 122.932^\circ$ \]

$\Rightarrow$ Galactic coordinates of Sgr A* are $[l, b] = (0.9444^\circ, -0.0461^\circ)$

Sgr A* is not at $(l, b) = (0, 0)$ leading to a "distance discrepancy" $d = \Delta \theta \times R$. 

\[
\Delta \theta = \left[(\Delta l)^2 + (\Delta b)^2\right]^{1/2} = 0.9455^\circ = 0.0165 \text{ rad}
\]

Distance discrepancy $d = (0.0165) \times 8.5 \text{ kpc} \approx 140 \text{ pc}$

or 10 pc for $l_{NCP} = 122.932^\circ$

One way to calculate the maximum time Sgr A* can be observed from Keck is to solve the spherical triangle ABC shown in figure (1) attached at the end of this solution set. Sgr A* is assumed to transit at the time shown in the figure, A is the North Celestial Pole, B is zenith with RA equal to Sgr A*'s RA. C marks the point on Sgr A*'s apparent path where it reaches Keck's elevation limit. We
want to calculate the time Sgr A* remains observable as its apparent motion
takes it from the elevation limit to the point of highest elevation (transit) and
to the other elevation limit, i.e. the time equivalent of the RA Angle A for both
elevation limits.

The sides of the triangle are
\[ a = 90 - \alpha \text{ where } \alpha \text{ is the elevation cutoff for Keck} \]
\[ b = 90 - \delta_{Sgr A^*} \]
\[ c = 90 - \text{latitude(Keck)} \]
Spherical trigonometry: \( \cos a = \cos b \cos c + \sin b \sin c \cos A \) or
\[
\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos (90 - \alpha) - \cos (90 - \delta) \cos (90 - \text{lat})}{\sin (90 - \delta) \sin (90 - \text{lat})}
\]
\[
= \frac{\sin \alpha - \sin \delta \sin (\text{lat})}{\cos \delta \cos (\text{lat})}
\]

Latitude = 19.7817°, for \( \alpha = 18^\circ \) West elevation limit we get \( A = 54.91^\circ \) or
3.661 hours and for \( \alpha = 33.3^\circ \) East elevation limit we get \( A = 29.94^\circ \) or
1.996 hours. Hence maximum time Sgr A* can be observed is (3.661+1.996) =
5.657 hours or 5 hr 40 min

The optimum time of year when we could actually hope to observe for that
long is when Sgr A* is near opposition, i.e when \( \alpha_{Sgr A^*} = 17h \text{ 45m} \) transits at
midnight i.e. in the middle of June

Problem 2 - Proper Motion

We are given the proper motion components (per year, let’s assume) \( \mu_\alpha =
-0.0374'' \) and \( \mu_\delta = 1.21'' \) of a star at declination \( \delta = 42.95^\circ \). When calculating
proper motion components it’s important to remember that they depend on
how \( \mu_\alpha \) has been derived. If it’s the distance in arcsec our star has moved in
successive images of the same field then this measured \( \mu_\alpha \) is indeed true stellar
proper motion along the RA axis. However, if we assume \( \mu_\alpha \) is found from
successive RA positions (remember iso-RA lines converge) of the star then

\[ \mu_{\alpha \text{ true}} = \mu_\alpha \cos \delta = -0.0374'' \times \cos (42.95)^\circ = -0.02735'' \]

Total proper motion

\[
\mu = \sqrt{(\mu \sin \theta)^2 + (\mu \cos \theta)^2} = \sqrt{(\mu_{\alpha \text{ true}})^2 + (\mu_\delta)^2} = 1.21''
\]
\[
\theta = \arctan \frac{\mu_{\alpha \text{ true}}}{\mu_\delta} = \arctan (-0.022624) = -1.30^\circ
\]

Total proper motion \( |\mu| = 1.21'' \), \( \theta = 1.30^\circ \)
The measured blueshift is \( v_{\text{rad}} = 7.6 \text{ km/s} \) and parallax is \( \phi = 0.376" \). The distance to the star is

\[
d = \frac{1 \text{ pc}}{\phi} = 2.66 \text{ pc}
\]

\( v_{\text{trans}} \), proper motion \( \times \) distance \( = \frac{1.21}{206265} \times 3.0857 \times 10^{13} \text{ km/yr} = 15.3 \text{ km/s} \)

Space velocity \( v = \sqrt{v_{\text{rad}}^2 + v_{\text{trans}}^2} = 17.1 \text{ km/s} \)

angle \( \psi = \arctan \frac{v_{\text{trans}}}{v_{\text{rad}}} = 63.6^\circ \)

where angle \( \psi \) is the angle between radial velocity vector and space velocity vector.

\[
v = 17.1 \text{km/s} , \quad \psi = 63.6^\circ
\]

At the time of closest approach, \( v = v_{\text{trans}} \), distance \( d_{\text{min}} = d \times \sin \psi = 2.38 \text{ pc} \) and the proper motion

\[
\mu_{\text{closest}} = \frac{v}{d_{\text{min}}} = \frac{17.1 \text{ km/s}}{2.38 \text{ pc}} = \frac{17.1 \times 3.156 \times 10^7 \times 206265}{2.38 \times 3.0857 \times 10^{13}} \text{ "/yr}
\]

\[
\mu_{\text{closest}} = 1.52 \text{ "/yr}
\]

\[\text{Problem 3 - Absolute Magnitude and Surface Brightness of the Galaxy}\]

We are told that the disk and bulge components of the Galaxy have luminosities \( L_{\text{disk}} = 1.2 \times 10^{10} L_\odot \) and \( L_{\text{bulge}} = 1.9 \times 10^9 \). Assuming the Galaxy, on average, is composed of stars much like our Sun,

\[
(M_B)_{M_\odot} - (M_B)_{gal} = 2.5 \log \frac{L_{gal}}{L_\odot}
\]

\[
(M_B)_{gal} = 5.48 - 2.5 \log (1.39 \times 10^{10}) = -19.88
\]

Now, the integrated luminosity of the bulge is 16% of the disk, plus we know that the disk is much bluer than the bulge especially the further out you go, so I think it’s also fine to approximate \( (L_B)_{gal} \approx (L_B)_{disk} \) which yields

\[
(M_B)_{gal} \approx -19.7
\]

From a fit to near-IR Galactic data we get a central disk luminosity density of \( 1208 L_\odot \text{ pc}^{-2} \) with an exponential disk scalelength of \( R_d = 2.7 \text{ kpc} \). At a
distance of $R = 8$ kpc, the local luminosity density falls by $e^{-8/2.7}$ to $\approx 62.4 L_\odot \text{pc}^{-2}$.

To calculate the projected surface brightness of the Galactic disk in the solar neighborhood let’s assume a disk of negligible thickness, viewed at a distance $d$ and inclination angle $i$ where this angle is measured between line of sight and vector perpendicular to the disk.

The flux observed from 1 arcsec squared $= \frac{L_{\odot} \text{from 1 arcsec}^2}{4\pi d^2}$. The area that subtends 1 arcsec squared is $\frac{d^2}{(206265)^2 \cos i}$ which means the flux is

$$f = \frac{62.4 L_\odot \times (3 \times 10^{18})^{-2} d^2}{(206265)^2 \cos i \times 4\pi d^2} = \frac{1.297 \times 10^{-47}}{\cos i} L_\odot \text{cm}^{-2}$$

Solar flux at 10 pc $f = \frac{L_\odot}{4\pi (3 \times 10^{18})^2} = 8.84 \times 10^{-41} L_\odot \text{cm}^{-2}$

$$(M_K)_{\text{gal}} - (M_K)_{\text{sun}} = -2.5 \log \frac{1.297 \times 10^{-47}}{8.84 \times 10^{-41} \times \cos i}$$

Using $(M_K)_{\text{sun}} = +3.28$ we get,

$$\mu_K = 20.36 + 2.5 \log (\cos i)$$

**Problem 4 - Malmquist Bias**

We wish to show that for a Gaussian Luminosity Function, $(M_0 - \langle M \rangle) = 0.6 \ln 10 \sigma^2$. (Note there is a typo in the problem set: 0.5 should be 0.6.) The complete derivation of this formula can be found in Binney and Merrifield, section 3.6.1, so I am not going to reproduce it here.

If $\sigma = 0.4$, the size of this effect is

$$\langle (M) - M_0 \rangle = -0.6 \ln 10 \sigma^2 = -0.22$$

For distance modulus $m - M_0 = 15$, the mean distance will be:

$$(m - \langle M \rangle) = (m - M_0) - \langle (M) - M_0 \rangle = 15.22 = 5 \log d - 5$$

$$d = 11 \text{kpc}$$

For a spatially uniformly distributed population of stars, the differential number counts of stars per unit area and per magnitude interval would take the form:

$$A(m) \sim \exp(0.6 \ln 10 (m - M_0))$$

(Again refer to Binney and Merrifield, section 3.6.1, page 114.)
Figure (1): Spherical triangle on celestial sphere to calculate angles when observing Sgr A* from Keck.