1. Salpeter’s Initial Mass Function (IMF) is of the form:

\[ \Phi(M) \propto M^{-(1+x)} \]

By considering only stars more massive than 1 solar mass (whose lifetimes are shorter than the age of the Galaxy) and stellar luminosities \( L \propto M^4 \), find the slope \( x \) such that equal numbers of stars are seen in a homogeneous isotropic region within equal logarithmic ranges of luminosity. What type of star dominates the counts if the slope \( x \) is flatter than this value?

2. The star formation history of a stellar population is often represented by an exponential decay from an initial burst, viz: \( \Psi(t) \propto \exp(-t/\tau) \) where \( \tau \) is some time constant. If the IMF \( \Phi(M) \) is invariant, obtain an expression for the observed number of stars of a given mass at time \( t \) in terms of its main sequence lifetime. Comment briefly on the differences you would expect to see in the H-R diagrams of a population where \( \tau = 0.1 \) Gyr and \( \tau = \infty \) for a population viewed after 12 Gyr.

For a population formed instantaneously with a Salpeter IMF and an upper mass cut-off at \( 2 M_\odot \), estimate the time after which most light comes from post-main sequence stars. Assume the time a star spends on the main sequence is \( \propto M/L \).

3. Prove that if a homogeneous sphere of a pressureless fluid with density \( \rho \) is released from rest, it will collapse to a point in a freefall time \( (3\pi/32G\rho)^{1/2} \).

4. Suppose a star is moving on a purely radial orbit in the potential \( \Phi = (2\pi G \rho/3)r^2 \), where the density \( \rho \) is a constant. Write down an equation for the acceleration of the star and hence show it undergoes simple harmonic motion about the center with period \( (3\pi G \rho)^{1/2} \).

5. Show that the potential generated by the spherical density distribution

\[ \rho(r) = (M/4\pi a^3) \left[ a^4 / (r^2 (r+a)^2) \right] \]

is:

\[ \Phi(r) = (GM/a) \ln \left[ r / (r+a) \right] \]

where \( M \) and \( a \) are constants. Verify that the total mass of the system is \( M \). Show that the circular speed is constant at small radii \( r \ll a \) and declines as \( r^{-1/2} \) at large radii.