This is a closed book exam. No notes (not even your lecture notes) or consultation. An unprogrammable handheld calculator is acceptable, if you need it. Please take three straight hours in a quiet, isolated, location and do not look at the problems until you are ready to begin. All questions carry equal weight.

Please embed clear written explanations inbetween your equations! I will penalize solutions whose logic I can’t follow.

Some possibly useful constants are provided at the end of the questions.

Please hand in your completed paper to Judy McClain in person by Friday December 11th at 9.00 am. This is a hard deadline as I leave the country that afternoon.

**Attempt FIVE questions**

1. For an assumed equation of state, a solution for the run of pressure, temperature and density can be obtained for a star in hydrostatic equilibrium. Show that, for a polytropic equation of state:

   \[ P = K \rho^{\gamma} \]

   where \( K \) and \( \gamma \) are independent of radius, the density dependence is given by the Lane-Emden equation

   \[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \]

   where the density and radius are expressed in terms of dimensionless variables via \( \rho = \lambda \theta^n \), the radius \( r = \alpha \xi \), respectively and the index \( n = (\gamma - 1)^{-1} \).

   Accordingly, show that the gravitational potential energy of a star is then given by the expression

   \[ \Omega = -\frac{3GM^2}{(5-n)R} \]
What is the implication of this result for the range of $n$ applicable to stable stars. Summarize the situations during stellar evolution for which polytropic solutions are appropriate, explaining clearly the value of $n$ that is applicable in each case.

2. Show that, for the general case, the pressure in an electron gas is given by the integral of the kind

$$P = \frac{1}{3} \int_{0}^{p_F} v_p f(p) \, dp = \frac{8\pi c}{3h^3} \int_{0}^{p_F} \frac{p / (m_e c)}{\sqrt{1 + p^2 / (m_e c^2)}} \, dp$$

where $m_e$ is the electron rest mass and $p_F$ the Fermi momentum.

Now consider how this expression can be simplified in the non-relativistic and ultra-relativistic case. Derive the critical density, $\rho_{\text{crit}}$, for the transition between these two extremes assuming, for simplicity, that $n_e = \rho / m_H$ in determining the electron number density.

Explain, in simple terms (i.e. without further lengthy derivations), why the radius of a degenerate non-relativistic white dwarf becomes smaller as its mass increases and why, in the ultrarelativistic case, there is a maximum stable mass.

3. Prove that the sound speed $c_S$ for an isothermal gas of uniform density $\rho$ is given by the expression:

$$c_S = \frac{\gamma k T}{\mu m_H}$$

where $\mu$ is the mean molecular weight. Provide an order of magnitude estimate of the value for the Sun. What is meant by the scale height $H$ of the solar atmosphere?

Discuss qualitatively how pressure waves can propagate within the Sun provided their angular frequency $\omega$ exceeds a critical value $\omega_C = c_S / 2H$. Estimate the value of $\omega_C$ just below the solar photosphere.

Summarize what has been learned from quantitative studies of pressure waves within the Sun.

4. Discuss quantitatively what is meant by ‘opacity’ in considering radiative transfer in a stellar interior. How is the term defined and what are the relevant units? What are the dominant contributors to opacity in the envelopes (i.e. the regions outside the thermonuclear core) of (i) a $1 M_\odot$ star, (ii) a $2.5 M_\odot$ star and (iii) a $5 M_\odot$ star?

In the outer layers of a star, assume $L(r) = L$ and $M(r) = M$, i.e. constants. Assume the opacity $\kappa$ is a power law in pressure and temperature, viz:

$$\kappa = \kappa_0 P^{\alpha-1} T^{4-\beta}$$

where $\kappa_0$, $\alpha$ and $\beta$ are constants. Under these conditions, derive a differential equation and boundary condition that will serve to determine $T(P)$ in the outer layers. Investigate the behavior of $T(P)$ at great depth and show that the outer layers must be convective if
\[ \alpha > \frac{2}{5} \beta > 0 \]

taking the adiabatic gradient to be \( \frac{\ddot{x}}{x} \).

5. Adopting the normal convention that a gas contains mass fractions \( X, Y \) and \( Z \) of hydrogen, helium and heavier metals, respectively, show clearly, indicating all the necessary arguments, that the mean molecular weight of a fully-ionized gas is given by:

\[
\mu = \frac{4}{(6X + Y + 2)}
\]

Simplify this result for the case when the gas comprises only hydrogen and helium.

For a 1 \( M_\odot \) star converting hydrogen to helium, homology arguments indicate that the main sequence luminosity \( L \) and effective temperature \( T_{\text{eff}} \) scale with the mean molecular weight \( \mu \) according to:

\[
L \propto \mu^{7.8}; \quad T_{\text{eff}} \propto \mu^{2.2}
\]

Use this to sketch on a H-R diagram the motion of a solar mass star during its lifetime on the main sequence.

6. Calculate the mean thermal energy of a proton in the core of a star where hydrogen is burning at a temperature of \( 10^7 \) K. Estimate the closest distance such energy could bring two protons in proximity and compare this with the radius of each proton. Explain how this Coulomb barrier is overcome and (without any lengthy derivation) why, for non-resonant reactions, there is a preferred energy at which fusion occurs. Sketch how the binding energy/nucleon varies with atomic mass. Above temperatures of \( 3 \times 10^9 \) K, photons have sufficient energy to dissociate a heavy nucleus. Write down the expression for the relative abundances attained by species \( i, j \) and \( k \) where forward and backward reactions of the type \( i + j \rightarrow k \) and \( k \rightarrow i + j \) achieve statistical equilibrium.

Why does such a statistical equilibrium lead to the continued production of heavier elements at the time of Silicon burning, but as we proceed to the iron peak itself, the bias towards heavier products ends?

7. State the two physical differences between the behavior of a degenerate neutron gas and that of a degenerate electron gas. How do these affect the upper mass limit that we can expect for a neutron star? Estimate the radius of a neutron star of mass 1.4 \( M_\odot \).

Estimate the total energy than can power a supernova if a collapsing core of 1.4 solar masses of pure \( \text{Fe}^{56} \) shrinks to the size of a neutron star.

Why have supernova of Type Ia been so much effective in cosmological distance measurements than Type II supernova given we have a much better physical understanding of the latter events? Explain how, with the increased aperture of TMT, it might be possible to confirm the cosmic acceleration with Type II supernovae? What measurements would need to be made?
8. Assume that the specific intensity of radiation is given by

\[ I = I_0 + I_1 \cos \theta \]

where \( I_0 \gg I_1 \) (a slightly anisotropic field). Calculate (i) the net monochromatic flux, (ii) the energy density \( U \), and (iii) the radiation pressure \( P_{\text{rad}} \). Show that \( P_{\text{rad}} = U/3 \). How will these expressions change if \( I = B \), the (isotropic) Planck function?

Why do we introduce the concept of a source function \( S_\nu \) in discussions of radiative transfer in a stellar atmosphere? Under what conditions is it important to distinguish between \( S_\nu \) and the Planck function \( B_\nu \)?

Show that if the source function \( S_\nu(\tau_\nu) \) varies with optical depth \( \tau_\nu \) in a linear fashion, that this linearity is preserved in a limb darkening expression. What use is such a result?

Proton mass: \( m_P = 1.67262 \times 10^{-24} \text{ g} \)
Proton radius: \( r_P = 1.2 \times 10^{-13} \text{ cm} \)
Electron mass: \( m_e = 9.11 \times 10^{-28} \text{ g} \)
Neutron mass: \( M_N = 1.67493 \times 10^{-24} \text{ g} \)
Radius of \( \text{Fe}^{56} \) nucleus \( \simeq 3.10^{-13} \text{ cm} \)
Planck’s constant: \( h = 6.6 \times 10^{-27} \text{ ergs s} \)
Solar mass: \( M_\odot = 1.99 \times 10^{33} \text{ g} \)
Solar luminosity: \( L_\odot = 3.85 \times 10^{33} \text{ ergs sec}^{-1} \)
Solar radius: \( R_\odot = 6.96 \times 10^{10} \text{ cm} \)
Boltzmann’s constant: \( k = 1.38 \times 10^{-16} \text{ erg deg}^{-1} \)
Stefan-Boltzmann constant: \( \sigma = 5.67 \times 10^{-5} \text{ ergs cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1} \)
Radiation density constant: \( a = 7.56 \times 10^{-15} \text{ ergs cm}^{-3} \text{ deg}^{-4} \)
Energy conversion: \( 1 \text{ eV} = 1.6 \times 10^{-12} \text{ ergs} \)