Supplement – Examples for Lecture V

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I. A WORKED EXAMPLE: VECTOR CALCULUS IN POLAR COORDINATES

In this section, we will do some examples from vector calculus in polar coordinates on $\mathbb{R}^2$. This is a simple case, but should be useful to exercise the machinery. Recall that the metric tensor components were

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad \text{and} \quad g_{r\theta} = g_{\theta r} = 0, \tag{1}$$

and the inverse metric is

$$g^{rr} = 1, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad \text{and} \quad g^{r\theta} = g^{\theta r} = 0. \tag{2}$$

The coordinate basis vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are not orthonormal, but we may define an orthonormal basis via

$$\boldsymbol{e}_{\hat{r}} = \boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\boldsymbol{e}_\theta. \tag{3}$$

A. Christoffel symbols

We begin by computing the Christoffel symbols for polar coordinates. The only nonzero derivative of a covariant metric component is

$$g_{\theta\theta,r} = 2r. \tag{4}$$

Now returning to the general rule,

$$\Gamma^\epsilon{}_{\delta\eta} = \frac{1}{2} g^{\epsilon\tau}(-g_{\delta\eta,\tau} + g_{\eta\tau,\delta} + g_{\delta\tau,\eta}), \tag{5}$$

we can directly read off the Christoffel symbols. They are:

$$\begin{aligned}
\Gamma^r{}_{rr} &= 0, \\
\Gamma^r{}_{\theta r} = \Gamma^r{}_{r\theta} &= 0, \\
\Gamma^r{}_{\theta\theta} &= \frac{1}{2} g^{rr}(-g_{\theta\theta,r}) = \frac{1}{2}(1)(-2r) = -r, \\
\Gamma^\theta{}_{rr} &= 0, \\
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{2} g^{\theta\theta}(g_{\theta\theta,r}) = \frac{1}{2}(r^{-2})(2r) = \frac{1}{r}, \quad \text{and} \\
\Gamma^\theta{}_{\theta\theta} &= 0.
\end{aligned} \tag{6}$$

So of the 6 Christoffel symbols, only 2 are nonzero. This is typical of highly symmetrical manifolds (expressed in coordinates that use the symmetry).

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B. Covariant derivative of a vector field

Let's now find the covariant derivative of a vector field $\boldsymbol{v}$. Using the rule from the last section,

$$v^{\alpha}{}_{;\beta} = v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\beta\gamma} v^{\gamma}. \tag{7}$$

Component by component, this reads

$$\begin{aligned} v^{r}{}_{;r} &= v^{r}{}_{,r}, \\ v^{r}{}_{;\theta} &= v^{r}{}_{,\theta} - r v^{\theta}, \\ v^{\theta}{}_{;r} &= v^{\theta}{}_{,r} + \frac{1}{r} v^{\theta}, \quad \text{and} \\ v^{\theta}{}_{;\theta} &= v^{\theta}{}_{,\theta} + \frac{1}{r} v^{r}. \end{aligned} \tag{8}$$

The divergence of a vector field is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = v^{r}{}_{;r} + v^{\theta}{}_{;\theta} = v^{r}{}_{,r} + v^{\theta}{}_{,\theta} + \frac{1}{r} v^{r}. \tag{9}$$

C. Examples of vector fields and their properties

Consider the vector field $\boldsymbol{v}$ that points in the 1-direction of the original Cartesian coordinate system ($\boldsymbol{v} = \boldsymbol{e}_1$). It can be expressed in terms of its components in the orthonormal basis

$$v^{\hat{r}} = \cos\theta, \quad v^{\hat{\theta}} = -\sin\theta; \tag{10}$$

or in the coordinate basis,

$$v^{r} = \cos\theta, \quad v^{\theta} = -\frac{1}{r}\sin\theta. \tag{11}$$

You know intuitively that this vector field is "constant," but that is not obvious in the polar coordinate system. We can still prove it, however, using Eq. (8):

$$\begin{aligned} v^{r}{}_{;r} &= v^{r}{}_{,r} = 0, \\ v^{r}{}_{;\theta} &= v^{r}{}_{,\theta} - r v^{\theta} = -\sin\theta - r(-\frac{1}{r}\sin\theta) = 0, \\ v^{\theta}{}_{;r} &= v^{\theta}{}_{,r} + \frac{1}{r} v^{\theta} = \frac{1}{r}^{2}\sin\theta + \frac{1}{r}(-\frac{1}{r}\sin\theta) = 0, \quad \text{and} \\ v^{\theta}{}_{;\theta} &= v^{\theta}{}_{,\theta} + \frac{1}{r} v^{r} = -\frac{1}{r}\cos\theta + \frac{1}{r}\cos\theta = 0. \end{aligned} \tag{12}$$

As a less trivial example, we can search for an axisymmetric radial vector field $\boldsymbol{E}$ ($E^{\theta} = 0$, E^{r} depends only on r and not θ) with zero divergence (except at the origin): $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$. Equation (9) tells us that we need

$$0 = E^{r}{}_{,r} + 0 + \frac{1}{r} E^{r}. \tag{13}$$

Since s^{r} depends on r, we may then write

$$\frac{d}{dr}(r E^{r}) = r E^{r}{}_{,r} + E^{r} = 0. \tag{14}$$

Therefore $E^{\hat{r}} = E^{r} \propto r^{-1}$. You may recognize this as the result that the electric field of a linear charge scales as $1/r$.