Ph236 Homework 5 Solutions

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Fall 2012

Problem 1: Circumference of a circle

a) Let us denote the trajectory of the geodesic $\mathcal{P}(\lambda, n)$ by $x^\alpha(\lambda, n)$, and so we have

$$v^\alpha(\lambda, n) = \frac{dx^\alpha(\lambda, n)}{d\lambda}, \tag{1}$$

and so

$$x^\alpha(\lambda, n) = \int_0^\lambda v^\alpha(\lambda', n) d\lambda', \tag{2}$$

since the above is the ordinary derivative and not the covariant derivative. Now $\vec{\xi}$ is defined as $\xi = \partial\mathcal{P}/\partial n$, hence

$$\xi^\alpha(\lambda, n) = \frac{\partial x^\alpha(\lambda, n)}{\partial n} = \frac{\partial}{\partial n}\int_0^\lambda v^\alpha(\lambda', n) d\lambda' = \int_0^\lambda \frac{\partial v^\alpha(\lambda', n)}{\partial n} d\lambda'. \tag{3}$$

So we immediately see that

$$\vec{\xi}(0, n) = \int_0^0 \frac{\partial v^\alpha(\lambda', n)}{\partial n} d\lambda' = 0. \tag{4}$$

We also find

$$\begin{aligned} \frac{D\vec{\xi}}{d\lambda} &= \frac{D\xi^\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda} = \frac{dx^\beta}{d\lambda}\xi^\alpha{}_{;\beta} = \frac{dx^\beta}{d\lambda}\left(\xi^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\mu}\xi^\mu\right) \\ &= \frac{dx^\beta}{d\lambda}\frac{\xi^\alpha}{dx^\beta} + \Gamma^\alpha{}_{\beta\mu}\frac{dx^\beta}{d\lambda}\xi^\mu = \frac{\xi^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\mu}v^\beta\xi^\mu, \end{aligned} \tag{5}$$

and therefore we get

$$\begin{aligned} \frac{D\vec{\xi}}{d\lambda}(0, n) &= \frac{d\xi^\alpha}{d\lambda}(0, n) + \Gamma^\alpha{}_{\beta\mu}(\mathcal{P}(0, n))v^\beta(0, n)\xi^\mu(0, n) = \frac{\partial v^\alpha(0, n)}{\partial n} + 0 \\ &= -\sin n\, a^\alpha + \cos n\, b^\alpha = \cos(n + \pi/2)\vec{a} + \sin(n + \pi/2)\vec{b} \\ &= \vec{v}(0, n + \pi/2), \end{aligned} \tag{6}$$

since $\xi^\mu(0,n) = 0$ and $v^\alpha(0,n) = \cos n\, a^\alpha = \sin n\, b^\alpha$. Note that the situation here, i.e. the meaning of $\vec{\xi}$ and $\vec{v}$ is identical to the situation considered in the section IV. GEODESIC DEVIATION of the lectures notes VI. Thus from the notes we know that

$$\frac{D^2\xi^\alpha}{d\lambda^2} = R^\alpha{}_{\beta\gamma\delta} v^\beta v^\gamma \xi^\delta, \tag{7}$$

and so

$$\frac{D^2\xi^\alpha}{d\lambda^2}(0,n) = R^\alpha{}_{\beta\gamma\delta}(\mathcal{P}(0,n)) v^\beta(0,n) v^\gamma(0,n) \xi^\delta(0,n) = 0, \tag{8}$$

since $\xi^\delta(0,n) = 0$. Finally, differentiating (7) yields

$$\begin{aligned}\frac{D^3\xi^\alpha}{d\lambda^3} &= \frac{DR^\alpha{}_{\beta\gamma\delta}}{d\lambda} v^\beta v^\gamma \xi^\delta + \frac{Dv^\beta}{d\lambda} R^\alpha{}_{\beta\gamma\delta} v^\gamma \xi^\delta + \frac{Dv^\gamma}{d\lambda} R^\alpha{}_{\beta\gamma\delta} v^\beta \xi^\delta \\ &\quad + \frac{D\xi^\delta}{d\lambda} R^\alpha{}_{\beta\gamma\delta} v^\beta v^\gamma \\ &= \frac{DR^\alpha{}_{\beta\gamma\delta}}{d\lambda} v^\beta v^\gamma \xi^\delta + \frac{D\xi^\delta}{d\lambda} R^\alpha{}_{\beta\gamma\delta} v^\beta v^\gamma, \end{aligned} \tag{9}$$

since $D\vec{v}/d\lambda = 0$, because the trajectories are geodesics. Evaluating the above at $\lambda = 0$ gives

$$\begin{aligned}\frac{D^3\xi^\alpha}{d\lambda^3}(0,n) &= \frac{DR^\alpha{}_{\beta\gamma\delta}}{d\lambda}(\mathcal{P}(0,n)) v^\beta(0,n) v^\gamma(0,n) \xi^\delta(0,n) \\ &\quad + v^\delta(0,n+\pi/2) R^\alpha{}_{\beta\gamma\delta}(\mathcal{P}(0,n)) v^\beta(0,n) v^\gamma(0,n) \\ &= -R^\alpha{}_{\beta\delta\gamma}(\mathcal{P}(0,n)) v^\beta(0,n) v^\delta(0,n+\pi/2) v^\gamma(0,n) \\ &= -\mathrm{Riemann}(\_, \vec{v}(0,n), \vec{v}(0,n+\pi/2), \vec{v}(0,n)), \end{aligned} \tag{10}$$

since $\xi^\delta(0,n) = 0$, since the Riemann tensor is antisymmetric in the last two indices and where we used (6). Thus we have shown all that we needed to show. □

b) We have $\psi = \vec{\xi}\cdot\vec{\xi} = \xi^\alpha \xi_\alpha$, but since this is a scalar, its covariant and ordinary partial derivatives are equal. Since we have already found the covariant derivatives of $\vec{\xi}$ at $\lambda = 0$ in part (a), it is actually advantageous to consider the covariant derivatives of ψ instead of the partial derivatives. We find

$$\left.\frac{\partial\psi}{\partial\lambda}\right|_{\lambda=0} = \left.\frac{D\psi}{d\lambda}\right|_{\lambda=0} = \left.2\xi^\alpha \frac{D\xi_\alpha}{d\lambda}\right|_{\lambda=0} = 0, \tag{11}$$

since $\xi^\alpha(0,n) = 0$. Using the above, we get

$$\begin{aligned}\left.\frac{\partial^2\psi}{\partial\lambda^2}\right|_{\lambda=0} &= \left.\frac{D^2\psi}{d\lambda^2}\right|_{\lambda=0} = 2\left.\frac{D}{d\lambda}\right|_{\lambda=0}\left(\xi^\alpha \frac{D\xi_\alpha}{d\lambda}\right) = 2\left.\left(\frac{D\xi^\alpha}{d\lambda}\frac{D\xi_\alpha}{d\lambda} + \xi^\alpha \frac{D^2\xi_\alpha}{d\lambda^2}\right)\right|_{\lambda=0} \\ &= 2\,|\vec{v}(0,n+\pi/2)|^2 + 0 = 2, \end{aligned} \tag{12}$$

since $\xi^\alpha(0,n)=0$ and where we used (6) and the fact that $\vec{v}(0,n+\pi/2)=-\sin n\,\vec{a}+\cos n\,\vec{b}$ has length 1 because $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors. Again using the above, we obtain

$$\left.\frac{\partial^3\psi}{\partial\lambda^3}\right|_{\lambda=0}=\left.\frac{D^3\psi}{d\lambda^3}\right|_{\lambda=0}=2\left.\frac{D}{d\lambda}\right|_{\lambda=0}\left(\frac{D\xi^\alpha}{d\lambda}\frac{D\xi_\alpha}{d\lambda}+\xi^\alpha\frac{D^2\xi_\alpha}{d\lambda^2}\right)$$
$$=2\left.\left(2\frac{D\xi^\alpha}{d\lambda}\frac{D^2\xi_\alpha}{d\lambda^2}+\frac{D\xi^\alpha}{d\lambda}\frac{D^2\xi_\alpha}{d\lambda^2}+\xi^\alpha\frac{D^3\xi_\alpha}{d\lambda^3}\right)\right|_{\lambda=0}=0, \tag{13}$$

since $\xi^\alpha(0,n)=0$ and $D^2\xi_\alpha/d\lambda^2=0$ at $\lambda=0$. Finally, using the above gives

$$\left.\frac{\partial^4\psi}{\partial\lambda^4}\right|_{\lambda=0}=\left.\frac{D^4\psi}{d\lambda^4}\right|_{\lambda=0}=2\left.\frac{D}{d\lambda}\right|_{\lambda=0}\left(3\frac{D\xi^\alpha}{d\lambda}\frac{D^2\xi_\alpha}{d\lambda^2}+\xi^\alpha\frac{D^3\xi_\alpha}{d\lambda^3}\right)$$
$$=2\left.\left(3\frac{D^2\xi^\alpha}{d\lambda^2}\frac{D^2\xi_\alpha}{d\lambda^2}+3\frac{D\xi^\alpha}{d\lambda}\frac{D^3\xi_\alpha}{d\lambda^3}+\frac{D\xi^\alpha}{d\lambda}\frac{D^3\xi_\alpha}{d\lambda^3}+\xi^\alpha\frac{D^4\xi_\alpha}{d\lambda^4}\right)\right|_{\lambda=0}$$
$$=8\frac{D\xi^\alpha}{d\lambda}(0,n)\frac{D^3\xi_\alpha}{d\lambda^3}(0,n)$$
$$=8v^\alpha(0,n+\pi/2)R_{\alpha\beta\gamma\delta}(\mathcal{P}(0,n))v^\beta(0,n)v^\gamma(0,n)v^\delta(0,n+\pi/2)$$
$$=8\text{Riemann}(\vec{v}(0,n+\pi/2),\vec{v}(0,n),\vec{v}(0,n),\vec{v}(0,n+\pi/2)), \tag{14}$$

since $\xi^\alpha(0,n)=0$ and $D^2\xi_\alpha/d\lambda^2=0$ at $\lambda=0$ and where we used (10). Note that the Riemann tensor is evaluated at the origin $\mathcal{O}$. Now recall that

$$\vec{v}(0,n)=\cos n\,\vec{a}+\sin n\,\vec{b}$$
$$\vec{v}(0,n+\pi/2)=-\sin n\,\vec{a}+\cos n\,\vec{b}, \tag{15}$$

and also recall that the Riemann tensor is antisymmetric in the first two and last two indices, thus we have

$$\text{Riemann}(\vec{a},\vec{a},\_,\_)=-\text{Riemann}(\vec{a},\vec{a},\_,\_)=0 \tag{16}$$

and similarly

$$\text{Riemann}(\_,\_,\vec{a},\vec{a})=0 \tag{17}$$

for any vector $\vec{a}$. Since the Riemann tensor is a tensor, it is linear in all arguments. Thus we find

$$\left.\frac{\partial^4\psi}{\partial\lambda^4}\right|_{\lambda=0}$$
$$=8\Big[(-\sin^2 n\,\cos^2 n)\text{Riemann}(\vec{a},\vec{b},\vec{a},\vec{b})+\sin^4 n\,\text{Riemann}(\vec{a},\vec{b},\vec{b},\vec{a})$$
$$+\cos^4 n\,\text{Riemann}(\vec{b},\vec{a},\vec{a},\vec{b})+(-\sin^2 n\,\cos^2 n)\text{Riemann}(\vec{b},\vec{a},\vec{b},\vec{a})\Big]$$
$$=8\text{Riemann}(\vec{a},\vec{b},\vec{a},\vec{b})\left(-2\sin^2 n\,\cos^2 n-\sin^4 n-\cos^4 n\right)$$
$$=-8\text{Riemann}(\vec{a},\vec{b},\vec{a},\vec{b})(\sin^2 n+\cos^2 n)^2$$
$$=-8\text{Riemann}(\vec{a},\vec{b},\vec{a},\vec{b}). \tag{18}$$

c) We can expand $\psi = \vec{\xi} \cdot \vec{\xi} = |\vec{\xi}|^2$ as a Taylor series about $\lambda = 0$. This gives

$$\begin{aligned}
|\vec{\xi}|^2 &= \psi(0) + \lambda \frac{\partial \psi}{\partial \lambda}(0) + \frac{\lambda^2}{2}\frac{\partial^2 \psi}{\partial \lambda^2}(0) + \frac{\lambda^3}{6}\frac{\partial^3 \psi}{\partial \lambda^3}(0) + \frac{\lambda^4}{24}\frac{\partial^4 \psi}{\partial \lambda^4}(0) + O(\lambda^5) \\
&= 0 + 0 + \frac{\lambda^2}{2}(2) + 0 + \frac{\lambda^4}{24}\left(-8\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b})\right) + O(\lambda^5) \\
&= \lambda^2 - \frac{\lambda^4}{3}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\lambda^5). \qquad (19)
\end{aligned}$$

Now expanding $|\vec{\xi}| = \sqrt{|\vec{\xi}|^2}$ around $\lambda = 0$ yields

$$|\vec{\xi}| = \lambda - \frac{\lambda^3}{6}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\lambda^4), \qquad (20)$$

since $\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b})$ is a constant, because the Riemann tensor is evaluated at the origin $\mathcal{O}$.

Recall that the circle is described by $\mathcal{C}(n) = \mathcal{P}(\rho, n)$, where n is the parameter ranging from 0 to 2π. The circumference of the circle is simply the length of the curve $\mathcal{C}(n)$ from $n = 0$ to $n = 2\pi$, which is given by

$$\begin{aligned}
C &= \int_0^{2\pi} |\mathcal{C}'(n)| dn = \int_0^{2\pi} \left| \frac{\partial \mathcal{P}(\rho, n)}{\partial n} \right| dn = \int_0^{2\pi} \left| \vec{\xi}(\rho, n) \right| dn \\
&= \int_0^{2\pi} \left(\rho - \frac{\rho^3}{6}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\rho^4) \right) dn \\
&= \left(\rho - \frac{\rho^3}{6}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\rho^4) \right) \int_0^{2\pi} dn \\
&= \left(\rho - \frac{\rho^3}{6}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\rho^4) \right) (2\pi) \\
&= 2\pi\rho - \frac{\pi\rho^3}{3}\text{Riemann}(\vec{a}, \vec{b}, \vec{a}, \vec{b}) + O(\rho^4), \qquad (21)
\end{aligned}$$

Problem 2: Geodetic precession

a)

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt}\frac{dt}{d\tau} = \dot{x}^\alpha u^0. \qquad (22)$$

Since $\vec{u} \cdot \vec{l} = 0$, we find

$$0 = \vec{u} \cdot \vec{l} = u^\alpha l_\alpha = u^0 \cdot x^\alpha l_\alpha = u^0 \dot{x}^0 l_0 + u^0 \dot{x}^i l_i, \qquad (23)$$

but $\dot{x}^0 = dt/dt = 1$ and so

$$\begin{aligned}
u^0 \dot{x}^0 l_0 &= u^0 l_0 = u^0 g_{0\mu} l^\mu = u^0 g_{00} l^0 = -(1 + 2\Phi) u^0 l^0 = -u^0 l^0 - 2\Phi u^0 l^0 \\
&= -u^0 l^0 + O(\epsilon^{3/2}), \qquad (24)
\end{aligned}$$

since the metric g is diagonal and since $\Phi \sim O(\epsilon)$ and $u^0 \sim O(\epsilon^{1/2})$. So to order ϵ, (23) becomes

$$\begin{aligned} 0 &= -u^0 l^0 + u^0 \dot{x}^i l_i \\ \Leftrightarrow \; l^0 &= \dot{x}^i l_i = \dot{x}_i l^i, \end{aligned} \tag{25}$$

b) Note that we can write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} - 2\Phi\delta_{\mu\nu}, \tag{26}$$

where $h_{\mu\nu} = -2\Phi\delta_{\mu\nu}$. Since the background metric is the Minkowski metric, we have $x_i = x^i$ and so we will raise and lower spatial indices without mentioning it throughout the remainder of this problem. The Christoffel symbols with a spatial index as the first index are given by

$$\begin{aligned} \Gamma^i{}_{\alpha\beta} &= \frac{1}{2} g^{i\mu} \left(-g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha}\right) = \frac{1}{2}(-h_{\alpha\beta}{}^{,i} + h^i_{\alpha,\beta} + h^i_{\beta,\alpha}) \\ &= \delta_{\alpha\beta}\Phi^{,i} - \delta^i{}_\alpha \Phi_{,\beta} - \delta^i{}_\beta \Phi_{,\alpha}. \end{aligned} \tag{27}$$

Since the vector $\vec{l}$ is parallel transported, for all i we have

$$\begin{aligned} 0 = \frac{Dl^i}{dt} &= \frac{Dl^i}{dx^\alpha}\dot{x}^\alpha = \frac{dl^i}{dx^\alpha}\dot{x}^\alpha + \Gamma^i{}_{\alpha\beta} l^\beta \dot{x}^\alpha = \frac{dl^i}{dt} + \Gamma^i{}_{\alpha\beta} l^\beta \dot{x}^\alpha \\ \Leftrightarrow \; \frac{dl^i}{dt} &= -\Gamma^i{}_{\alpha\beta} l^\beta \dot{x}^\alpha. \end{aligned} \tag{28}$$

Carefully evaluating $-\Gamma^i{}_{\alpha\beta} l^\beta \dot{x}^\alpha$ yields

$$\begin{aligned} \frac{dl^i}{dt} &= -\Gamma^i{}_{\alpha\beta} l^\beta \dot{x}^\alpha = -\Gamma^i{}_{00} l^0 \dot{x}^0 - \Gamma^i{}_{0j} l^j \dot{x}^0 - \Gamma^i{}_{k0} l^0 \dot{x}^k - \Gamma^i{}_{jk} l^k \dot{x}^j \\ &= -l^0 \dot{x}^0 (\Phi^{,i} - 0 - 0) - l^j \dot{x}^0 (0 - 0 - \delta^i{}_j \dot{\Phi}) - l^0 \dot{x}^k (0 - \delta^i{}_k \dot{\Phi} - 0) \\ &\quad - l^k \dot{x}^j (\delta_{jk}\Phi^{,i} - \delta^i{}_j \Phi_{,k} - \delta^i{}_k \Phi_{,j}) \\ &= -l^0 \Phi^{,i} + l^i \dot{\Phi} - l^0 \dot{x}^i \dot{\Phi} - l^k \dot{x}_k \Phi^{,i} + l^k \dot{x}^i \Phi_{,k} + l^i \dot{x}^j \Phi_{,j}. \end{aligned} \tag{29}$$

But now recall that $l^0 = l^k \dot{x}_k$, and so the above becomes

$$\frac{dl^i}{dt} = -2l^k \dot{x}_k \Phi^{,i} + l^i \dot{\Phi} + l^k \dot{x}^i \Phi_{,k} + l^i \dot{x}^j \Phi_{,j} - l^0 \dot{x}^i \dot{\Phi}. \tag{30}$$

Recall that $\Phi \sim O(\epsilon)$, $\dot{x}^\mu \sim O(\epsilon^{1/2}/t_{\rm dyn})$, $\dot{\Phi} \sim O(\epsilon/t_{\rm dyn})$ and finally

$$\Phi_{,i} = \frac{d\Phi}{dx^i} \sim \frac{O(\epsilon)}{O(\epsilon^{1/2})} = O(\epsilon^{1/2}). \tag{31}$$

So $\dot{x}^i \Phi_{,k} \sim O(\epsilon^{1/2}/t_{\rm dyn}) O(\epsilon^{1/2}) = O(\epsilon/t_{\rm dyn})$, which means that the first, third and fourth terms are of order $\epsilon/t_{\rm dyn}$. The second term is also of this order because $\dot{\Phi} \sim O(\epsilon/t_{\rm dyn})$, but the last term goes as $\dot{x}^j\dot{\Phi} \sim O(\epsilon^{1/2}/t_{\rm dyn})O(\epsilon/t_{\rm dyn}) = O(\epsilon^{3/2}/t^2_{\rm dyn})$ and so we drop this term. Thus to order $\epsilon/t_{\rm dyn}$ we found that

$$\frac{dl^i}{dt} = -2l^k\dot{x}_k\Phi^{,i} + l^i\dot{\Phi} + l^k\dot{x}^i\Phi_{,k} + l^i\dot{x}^j\Phi_{,j}, \tag{32}$$

c) By using the definition of the time average and integration by parts we find

$$\begin{aligned}\langle \dot{x}^i\ddot{x}^j\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{x}^i\ddot{x}^j\,dt = \lim_{T\to\infty}\frac{1}{T}\left(\left(\dot{x}^i\dot{x}^j\right)\Big|_0^T - \int_0^T \ddot{x}^i\dot{x}^j\,dt\right) \\ &= 0 - \lim_{T\to\infty}\frac{1}{T}\int_0^T \ddot{x}^i\dot{x}^j\,dt = -\langle \ddot{x}^i\dot{x}^j\rangle = -\langle \dot{x}^j\ddot{x}^i\rangle\,,\end{aligned} \tag{33}$$

since we assume that $\dot{x}^i(T)\dot{x}^j(T)$ remains finite as $T\to\infty$. Thus we have shown that $\langle \dot{x}^i\ddot{x}^j\rangle$ is antisymmetric in i and j. □

The Newtonian equation of motion is simply $\ddot{x}^j = -\Phi^{,j}$ and so we have

$$\langle \dot{x}^i\Phi_{,j}\rangle = -\langle \dot{x}^i\ddot{x}_j\rangle = -\langle \dot{x}_i\ddot{x}_j\rangle\,. \tag{34}$$

Recall that $B^k = \epsilon^{ijk}\langle \dot{x}_i\ddot{x}_j\rangle$ and so if we multiply both sides by ϵ_{mnk} we get

$$\begin{aligned}\epsilon_{mnk}B^k &= \epsilon_{mnk}e^{ijk}\langle \dot{x}_i\ddot{x}_j\rangle = (\delta^i{}_m\delta^j{}_n - \delta^j{}_m\delta^i{}_n)\langle \dot{x}_i\ddot{x}_j\rangle \\ &= \langle \dot{x}_m\ddot{x}_n\rangle - \langle \dot{x}_n\ddot{x}_m\rangle = 2\langle \dot{x}_m\ddot{x}_n\rangle\,.\end{aligned} \tag{35}$$

Thus we obtain

$$\langle \dot{x}^i\Phi_{,j}\rangle = -\langle \dot{x}_i\ddot{x}_j\rangle = -\frac{1}{2}\epsilon_{ijk}B^k. \tag{36}$$

d) Using the above, (32) gives

$$\begin{aligned}\left\langle\frac{dl^i}{dt}\right\rangle &= -2l^k\langle \dot{x}_k\Phi^{,i}\rangle + l^i\langle\dot{\Phi}\rangle + l^k\langle \dot{x}^i\Phi_{,k}\rangle + l^i\langle \dot{x}^j\Phi_{,j}\rangle \\ &= l^k\epsilon_{kil}B^l + 0 - \frac{1}{2}l^k\epsilon_{ikl}B^l - \frac{1}{2}l^i\epsilon_{jjl}B^l \\ &= \epsilon_{ilk}B^l l^k + \frac{1}{2}e_{ilk}B^l l^k - 0 = \frac{3}{2}\epsilon_{ijk}B^j l^k,\end{aligned} \tag{37}$$

since $\langle\dot{\Phi}\rangle = 0$.

e) Consider a circular equatorial orbit, with $r = R$, $\theta = \pi/2$ and $\phi = \Omega t$, where $\Omega = \sqrt{GM/R^3}$ is given by Kepler's law. Then we have

$$\begin{aligned}\vec{x}(t) &= R(\cos\Omega t, \sin\Omega t, 0) \\ \dot{\vec{x}}(t) &= R\Omega(-\sin\Omega t, \cos\Omega t, 0) \\ \ddot{\vec{x}}(t) &= -R\Omega^2(\cos\Omega t, \sin\Omega t, 0).\end{aligned} \tag{38}$$

So we find

$$\vec{B} = \dot{\vec{x}} \times \ddot{\vec{x}} = R^2\Omega^3(0,0,1) = R^2\Omega^3\hat{e}_z. \tag{39}$$

From (37) we see that $\vec{B}$ must have units of $\sec^{-1}$, so we need to divide the above by c^2. Thus we find

$$\vec{B} = \frac{R^2}{c^2}\left(\frac{GM}{R^3}\right)^{3/2}\hat{e}_z. \tag{40}$$

For mean radius of about $R = 6500$ km. The mass of the Earth is[2] $M_\odot = 6 \times 10^{24}$ kg. This gives

$$B \sim 6 \times 10^{-11} \text{ rad s}^{-1} \sim 5 \text{ arcsec yr}^{-1}. \tag{41}$$

Acknowledgments

We thank Jonas Lippuner for the use of his LaTeX template in writing these solutions.