Ph236 Homework 4 Solutions

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Problem 1: Gravitational "fields" and the equivalence principle

a) For $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$:

$$\begin{aligned}\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}(\eta^{\alpha\beta} + h^{\alpha\beta})(-h_{\mu\nu,\beta} + h_{\mu\beta,\nu} + h_{\nu\beta,\mu}) \\ &= \frac{1}{2}\eta^{\alpha\beta}(-h_{\mu\nu,\beta} + h_{\mu\beta,\nu} + h_{\nu\beta,\mu}) + O(h^2),\end{aligned} \tag{1}$$

Thus geodesic equation thus gives:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -\Gamma^{i}{}_{00}\left(\frac{dt}{dt}\right)^2 - 2\Gamma^{i}{}_{0j}\frac{dx^j}{dt}\frac{dt}{dt} - \Gamma^{i}{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &= -\Gamma^{i}{}_{00} + O\left(hv\right) + O\left(v^2\right) = -\Gamma^{i}{}_{00} = -\frac{1}{2}\eta^{ii}(-h_{00,i} + 2h_{0i,0}) \\ &= -\frac{\partial\Phi}{\partial x^i} - \frac{\partial A_i}{\partial t},\end{aligned} \tag{2}$$

to first order in h and v.

b) Since g is stationary,

$$\begin{aligned}\frac{d\tau}{dt} &= \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = \sqrt{-(\eta_{\mu\nu} + h_{\mu\nu})\dot{X}^\mu\dot{X}^\nu} \\ &= \sqrt{1 + 2\Phi - 2A_i\dot{X}^i - \dot{X}^i\dot{X}_i},\end{aligned} \tag{3}$$

since $\dot{X}^0 = 1$.

$\dot{x}^i_{\text{Alice}} = \dot{x}^i_{\text{Bob}} = 0$. Hence,

$$\frac{d\tau_{\text{Alice}}}{dt} = \sqrt{1 + 2\Phi(x_{\text{Alice}})} \tag{4}$$

$$\frac{d\tau_{\text{Bob}}}{dt} = \sqrt{1 + 2\Phi(x_{\text{Bob}})}, \tag{5}$$

and thus

$$\frac{d\tau_{\text{Alice}}}{d\tau_{\text{Bob}}} = \sqrt{\frac{1 + 2\Phi(x_{\text{Alice}})}{1 + 2\Phi(x_{\text{Bob}})}}. \tag{6}$$

$d\tau_{\text{Alice}}/d\tau_{\text{Bob}}$ is the gravitational redshift/blueshift that is observed by Alice for light emitted by Bob.

$d\tau_{\text{Alice}}/d\tau_{\text{Bob}}$ does not measure the absolute potential Φ.

c)

$$g_{00} = -(1+ax^3)^2 = -1 - 2ax^3 - a^2(x^3)^2 \approx -1 - 2ax^3, \tag{7}$$

since $|ax^3| \ll 1$. So if we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then $h_{00} = -2ax^3$ and $h_{0i} = h_{ij} = 0$. Comparing this to the h in part (a), we have $-2\Phi = -2ax^3$, hence $\Phi = ax^3$, and $A_{0i} = 0$. Thus the geodesics are

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i} = -\frac{\partial}{\partial x^i}\left(ax^3\right) = (0, 0, -a)\,. \tag{8}$$

From part (b) we find

$$\frac{d\tau}{dt} = \sqrt{1 + 2ax^3 - \left(\frac{dx^1}{dt}\right)^2 - \left(\frac{dx^2}{dt}\right)^2 - \left(\frac{dx^3}{dt}\right)^2}\,. \tag{9}$$

From the equation of a geodesic, we see that a is the magnitude of a constant force that a particle feels in the -3 direction.

d) Since our frame is accelerated with a constant acceleration a, we let our choice of coordinate transformation be inspired by the Rindler coordinates, and we define the primed coordinates as

$$\begin{aligned} t' &= \frac{1+ax^3}{a}\sinh(at), \\ x^{1'} &= x^1, \\ x^{2'} &= x^2, \\ x^{3'} &= \frac{1+ax^3}{a}\cosh(at). \end{aligned} \tag{10}$$

Instead of directly showing that we have the Minkowski metric in the primed coordinates, we will assume that we have the Minkowski metric there and then show that this leads exactly to the given metric in the unprimed coordinates. We have

$$\begin{aligned} ds'^2 &= -dt'^2 + (dx^{1'})^2 + (dx^{2'})^2 + (dx^{3'})^2 \\ &= -\left((1+ax^3)\cosh(at)dt + \sinh(at)dx^3\right)^2 + (dx^1)^2 + (dx^2)^2 \\ &\quad + \left((1+ax^3)\sinh(at)dt + \cosh(at)dx^3\right)^2 \\ &= -(1+ax^3)dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= ds^2, \end{aligned} \tag{11}$$

since $\cosh^2 x - \sinh^2 x = 1$.

Setting $x^1 = x^2 = x^3 = 0$ gives $t' = a^{-1}\sinh(at)$, $x^{1'} = 0$, $x^{2'} = 0$ and $x^{3'} = a^{-1}\cosh(at)$. Thus in the Minkowski spacetime coordinates, the trajectory is

$$x^{3'}(t') = \frac{1}{a}\cosh(at) = \frac{1}{a}\sqrt{1+\sinh^2(at)} = \frac{1}{a}\sqrt{1+a^2t'^2}. \tag{12}$$

The 4-velocity is

$$u^{\mu'} = \frac{dx^{\mu'}}{d\tau} = \frac{dt'}{d\tau}\frac{dx^{\mu'}}{dt'} = \frac{dt'}{d\tau}\left(1, 0, 0, \frac{at'}{\sqrt{1+a^2t'^2}}\right), \tag{13}$$

where

$$\begin{aligned} \frac{d\tau}{dt'} &= \sqrt{-\eta_{\mu'\nu'}\frac{dx^{\mu'}}{dt'}\frac{dx^{\mu'}}{dt'}} = \sqrt{1-\left(\frac{at'}{\sqrt{1+a^2t'^2}}\right)^2} = \sqrt{1-\frac{a^2t'^2}{1+a^2t'^2}} \\ &= \sqrt{\frac{1}{1+a^2t'^2}}, \end{aligned} \tag{14}$$

and so

$$u^{\mu'} = \sqrt{1+a^2t'^2}\left(1, 0, 0, \frac{at'}{\sqrt{1+a^2t'^2}}\right) = \left(\sqrt{1+a^2t'^2}, 0, 0, at'\right). \tag{15}$$

The 4-acceleration is

$$\begin{aligned} a^{\mu'} &= \frac{du^{\mu'}}{d\tau} = \frac{dt'}{d\tau}\frac{du^{\mu'}}{dt'} = \sqrt{1+a^2t'^2}\left(\frac{a^2t'}{\sqrt{1+a^2t'^2}}, 0, 0, a\right) \\ &= \left(a^2t', 0, 0, a\sqrt{1+a^2t'^2}\right), \end{aligned} \tag{16}$$

and so its magnitude is

$$\begin{aligned} |\vec{a}| &= \sqrt{a^{\mu'}a_{\mu'}} = \sqrt{\eta_{\mu'\nu'}a^{\mu'}a^{\nu'}} = \sqrt{-a^4t'^2 + a^2(1+a^2t'^2)} \\ &= \sqrt{-a^4t'^2 + a^2 + a^4t'^2} = \sqrt{a^2} = |a|, \end{aligned} \tag{17}$$

Problem 2: Orbits around a black hole

a) From the given form of ds^2 we immediately see that the metric tensor is diagonal (there are no cross-terms) and if we label $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, the components of the metric tensor are

$$\begin{aligned} g_{00} &= -\left(1 - \frac{2M}{x^1}\right) \\ g_{11} &= \frac{1}{1-2M/x^1} \\ g_{22} &= (x^1)^2 \\ g_{33} &= (x^1)^2\sin^2 x^2. \end{aligned} \tag{18}$$

Since g is diagonal, the components $g^{\alpha\beta}$ of the inverse metric are simply the reciprocals of the components $g_{\alpha\beta}$, hence

$$\begin{aligned} g^{00} &= -\frac{1}{1-2M/x^1} \\ g^{11} &= 1-\frac{2M}{x^1} \\ g^{22} &= \frac{1}{(x^1)^2} \\ g^{33} &= \frac{1}{(x^1)^2\sin^2 x^2}. \end{aligned} \tag{19}$$

The Christoffel symbols are given by

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(-g_{\mu\nu,\beta}+g_{\mu\beta,\nu}+g_{\nu\beta,\mu}) = \frac{1}{2}g^{\alpha\alpha}(-g_{\mu\nu,\alpha}+g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}), \tag{20}$$

since $g^{\alpha\beta}$ is diagonal. So for $\alpha=0$ we can only get a nonzero Christoffel symbol if one other index is 0, because only g_{00} is nonnegative, and the other index must be 1, because g_{00} only depends on x^1. Note that $g_{\mu\nu,0}=0$ for all μ and ν, because no component of the metric tensor depends on time. Thus the only nonzero Christoffel symbols with $\alpha=0$ are

$$\Gamma^0{}_{01} = \Gamma^0{}_{10} = \frac{M}{r^2-2Mr}. \tag{21}$$

For $\alpha=1$ we get nonzero Christoffel symbols when $\mu=\nu=0,1,2,3$, but $\mu=1$ and $\nu=0$ will give a zero Christoffel symbol because g_{11} does not depend on t. Thus the only nonzero Christoffel symbols with $\alpha=1$ are

$$\begin{aligned} \Gamma^1{}_{00} &= \frac{Mr-2M^2}{r^3} \\ \Gamma^1{}_{11} &= \frac{M}{2Mr-r^2} \\ \Gamma^1{}_{22} &= 2M-r \end{aligned} \tag{22}$$

and finally

$$\begin{aligned} \Gamma^1{}_{33} &= \frac{1}{2}\left(1-\frac{2M}{x^1}\right)(-2x^1\sin^2 x^2) \\ & 2M\sin^2\theta - r\sin^2\theta. \end{aligned} \tag{23}$$

For $\alpha=2$ we get nonzero Christoffel symbols for $\mu=\nu=3$ and for $\mu=2$ and $\nu=1$. Thus the only nonzero Christoffel symbols with $\alpha=2$ are

$$\Gamma^2{}_{12} = \frac{1}{r} \tag{24}$$

$$\Gamma^2{}_{33} = -\sin\theta\,\cos\theta. \tag{25}$$

Finally, for $\alpha = 3$ we can only get nonzero Christoffel symbols for $\mu = 1$ and $\nu = 3$, or $\mu = 2$ and $\nu = 3$, because no component of g depends on x^3 and g_{33} only depend on x^1 and x^2. Thus the only nonzero Christoffel symbols with $\alpha = 3$ are

$$\Gamma^3{}_{13} = \frac{1}{r} \tag{26}$$

$$\Gamma^3{}_{23} = \cot\theta. \tag{27}$$

b) We have

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt}\frac{dt}{d\tau}. \tag{28}$$

Since r and θ are constant, it immediately follows that $u^r = u^\theta = 0$. Since we are talking about a physical orbit of a particle, the 4-velocity must be time-like, thus we need

$$\begin{aligned} -1 = u^\alpha u_\alpha = g_{\alpha\beta}u^\alpha u^\beta &= -\left(1 - \frac{2M}{r}\right)(u^t)^2 + r^2\sin^2\theta\,(u^\phi)^2 \\ \Leftrightarrow\ r^2(u^\phi)^2 &= \left(1 - \frac{2M}{r}\right)(u^t)^2 - 1, \end{aligned} \tag{29}$$

since $\theta = \pi/2$.

c) Since the trajectory is a geodesic, we have

$$\begin{aligned} \frac{du^t}{d\tau} = \frac{d^2t}{d\tau^2} = \frac{d^2x^0}{d\tau^2} &= -\Gamma^0{}_{\mu\nu}u^\mu u^\nu = -\Gamma^0{}_{00}(u^t)^2 - \Gamma^0{}_{33}(u^\phi)^2 - 2\Gamma^0{}_{03}u^t u^\phi \\ &= 0, \end{aligned} \tag{30}$$

since $u^r = u^\theta = 0$ and all the remaining Chistoffel symbols are zero. Similarly,

$$\begin{aligned} \frac{du^\phi}{d\tau} = \frac{d^2\phi}{d\tau^2} = \frac{d^2x^3}{d\tau^2} &= -\Gamma^3{}_{\mu\nu}u^\mu u^\nu = -\Gamma^3{}_{00}(u^t)^2 - \Gamma^3{}_{33}(u^\phi)^2 - 2\Gamma^3{}_{03}u^t u^\phi \\ &= 0, \end{aligned} \tag{31}$$

since again $u^r = u^\theta = 0$ and the remaining the Christoffel symbols are all zero. Thus we have shown that u^t and u^ϕ are constants, since their time derivatives are zero. □

Since $u^r = 0$,

$$\begin{aligned} 0 = \frac{du^r}{d\tau} = \frac{d^2x^1}{d\tau^2} &= -\Gamma^1{}_{\mu\nu}u^\mu u^\nu = -\Gamma^1{}_{00}(u^t)^2 - \Gamma^1{}_{33}(u^\phi)^2 \\ \Leftrightarrow\ \frac{Mr - 2M^2}{r^3}(u^t)^2 &= -(2M - r)(u^\phi)^2 \\ \Leftrightarrow\ u^\phi &= \sqrt{\frac{M}{r^3}}u^t, \end{aligned} \tag{32}$$

since we take $u^\phi > 0$ and $u^t > 0$. We can now substitute the above into (29) to find

$$\frac{M}{r}(u^t)^2 = \left(1 - \frac{2M}{r}\right)(u^t)^2 - 1$$
$$\Leftrightarrow \quad u^t = \sqrt{\frac{r}{r-3M}}. \tag{33}$$

So for a given r, u^t is determined and then (32) also determines u^ϕ. (32) also gives

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{u^\phi}{u^t} = \sqrt{\frac{M}{r^3}}, \tag{34}$$

which is exactly equal to the Keplerian result.

Now consider a distant observer with local time t', that observer will actually see the angular frequency

$$\Omega' = \frac{d\phi}{dt'} = \frac{d\phi}{dt}\frac{dt}{dt'} = \Omega\frac{dt}{d\tau}\frac{d\tau}{dt'} = \Omega u^t\sqrt{1 - \frac{2M}{r}} = \Omega u^t, \tag{35}$$

since $r \to \infty$. Therefore, Ω actually corresponds to

$$\Omega = \frac{\Omega'}{u^t} = \Omega'\sqrt{\frac{r-3M}{r}} = \Omega'\sqrt{1 - \frac{3M}{r}}, \tag{36}$$

where we used (33) and where Ω' is the angular frequency observed by the distant observer, M is the mass of the black hole and r is the radius of the circular orbit of the particle. So Ω is actually the observed angular frequency times a correction factor that depends on the mass of the black hole and the radius of the orbit.

d) We see from (33) that there is no solution for u^t, and thus by extension for u^ϕ and a circular orbit, if $r \leq 3M$. Thus we can have circular orbits only for $r > 3M$.

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