

Ph236 Homework 3 Solutions

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1. Relativistic shocks

a) We are considering an ideal fluid with pre-shock flow parameters baryon number density n , energy density ρ , pressure p , and a 3-velocity β , and primed post-shock parameters. The number baryon flux is $S^\alpha = nu^\alpha$, where u^α is the 4-velocity of the fluid. We know that this baryonic flux is conserved across the shock, $\partial_\mu S^\mu = 0$. Due to the translational invariance in the x^2, x^3 directions the preshock 4-velocity is $u^\alpha = (\gamma, \gamma\beta, 0, 0)$, and equivalently the post-shock 4-velocity is $u'^\alpha = (\gamma', \gamma'\beta', 0, 0)$, in the rest frame of the shock. Conservation of baryonic flux thus implies that $nu^i = n'u'^i$, and we find the first conservation equation

$$n\gamma\beta = n'\gamma'\beta'. \quad (1)$$

We also impose the conservation of energy-momentum across the shock, $\partial_\mu T^{\mu\nu} = 0$. The energy-momentum tensor is written as $T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$. We note that the components of $T^{\mu\nu}$ are T^{00} , T^{0i} , and T^{ij} . Expanding the contraction in the conservation equation we see $\partial_\mu T^{\mu\nu} = \partial_0 T^{0\nu} + \partial_i T^{i\nu}$. Noting that the system is time invariant, in the rest frame of the shock we can therefore equate the pre-shock and post-shock T^{0i} and T^{ij} components. First, considering $T^{0i} = T'^{0i}$ we see that this implies $\rho u^0 u^i + p u^0 u^i = \rho' u'^0 u'^i + p' u'^0 u'^i$. Thus we find our second relation for the flow parameters,

$$(\rho + p)\gamma^2\beta = (\rho' + p')\gamma'^2\beta'. \quad (2)$$

Now equating $T^{ij} = T'^{ij}$, we see that $\rho u^i u^j + p(1 + u^i u^j) = \rho' u'^i u'^j + p'(1 + u'^i u'^j)$. Noting that $1 + \beta^2\gamma^2 = \frac{1}{1-\beta^2} = \gamma^2$, we find our third conservation equation,

$$(\rho\beta^2 + p)\gamma^2 = (\rho'\beta'^2 + p')\gamma'^2. \quad (3)$$

Thus we have found the three conservation equations for the flow parameters in the gas before and after the shock

$$\begin{aligned} n\beta\gamma &= n'\beta'\gamma' \\ (\rho + p)\beta\gamma^2 &= (\rho' + p')\beta'\gamma'^2 \\ (\rho\beta^2 + p)\gamma^2 &= (\rho'\beta'^2 + p')\gamma'^2 \end{aligned} \quad (4)$$

b) As the incoming gas is cold we may neglected thermal motion in the pre-shock energy density, i.e. $p \sim kT \ll \rho\beta^2$ which means we take the total energy density is $\rho = nm_0$, and comparatively we take $p \approx 0$. We know that the outgoing gas is radiation dominated. The equation of state for a radiation dominated gas is $p' = \frac{1}{3}\rho'_{\text{rad}}$, where ρ'_{rad} is the energy density from radiation and p' corresponds to the radiation pressure. The total energy density in the post-shock gas is thus $\rho' = 3p' + n'm_0$. We have found the three desired equations for p, ρ and ρ'

$$\rho = nm_0, \quad p = 0, \quad \rho' = 3p' + n'm_0. \quad (5)$$

c) Using the first two conservation equations and the conditions above 5, we find that

$$(\rho + p)\beta\gamma^2 = (\rho' + p')\beta'\gamma'^2 \rightarrow nm_0\beta\gamma^2 = (4p' + n'm_0)\beta'\gamma'^2 \quad (6)$$

$$p' = \frac{nm_0\beta\gamma^2}{4\beta'\gamma'^2} - \frac{nm_0\beta\gamma}{4\beta'\gamma'} = \frac{nm_0\beta\gamma}{4\beta'\gamma'^2}(\gamma - \gamma'). \quad (7)$$

d) We now use the last conservation equation in 4 together with the conditions on p, ρ and ρ' in 5, we find that

$$(\rho\beta^2 + p)\gamma^2 = (\rho'\beta'^2 + p')\gamma'^2 \rightarrow nm_0\beta^2\gamma^2 = (3\beta'^2\gamma'^2 + \gamma'^2)p' + nm_0\beta'\gamma'\beta\gamma \quad (8)$$

$$(3\beta'^2\gamma'^2 + \gamma'^2)\frac{\gamma - \gamma'}{4\beta'\gamma'^2} = \beta\gamma - \beta'\gamma' \rightarrow 4\beta = (3\beta'^2\gamma'^2 + \gamma'^2)\frac{\gamma - \gamma'}{\beta'\gamma'^2\gamma} + 4\frac{\beta'\gamma'}{\gamma} \quad (9)$$

$$4\beta = 3\beta' + \frac{1}{\beta'} - \frac{3\beta'\gamma'}{\gamma} - \frac{\gamma'}{\beta'\gamma} + 4\frac{\beta'\gamma'}{\gamma} = 3\beta' + \frac{1}{\beta'} - \frac{1}{\gamma\beta'}\frac{1 - \beta'^2}{\sqrt{1 - \beta'^2}}, \quad (10)$$

and thus we find the desired equation between the pre-shock and post-shock velocity

$$4\beta = 3\beta' + \frac{1}{\beta'} - \frac{\sqrt{1 - \beta'^2}}{\gamma\beta'}. \quad (11)$$

Looking at the equation we found, we should be able to spot one of the two solutions, or we can use MATHEMATICA to find the two solutions explicitly. The equation is trivially satisfied for the solution $\beta = \beta'$, which is not what we are interested in studying as nothing physically relevant happens. The nontrivial solution is given by

$$\beta = \frac{9\beta'^2 + 7}{15\beta'^2 + 1}\beta', \quad (12)$$

which is the solution we are interested in looking at.

e) The relation we found in part d is plotted in Figure 1. In the case of a non-relativistic shock, $\beta \rightarrow 0$. It should be clear from our plots that for the nontrivial solution $\beta' < \beta$. Thus, in the non-relativistic case where $\beta \rightarrow 0$, $\beta' \rightarrow 0$, as well, and we want to find the leading order behavior in both β and β' . It is clear from Eq. 12, that the leading order behavior is $\beta' = \frac{1}{7}\beta$. But we may also find this behavior from expanding Eq. 11,

$$4\beta\beta' = 3\beta'^2 + 1 - \left(1 - \frac{1}{2}\beta'^2 + \mathcal{O}(\beta'^4)\right) \left(1 - \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4)\right) \quad (13)$$

$$= \frac{7}{2}\beta'^2 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4, \beta'^4, \beta^2\beta'^2), \quad (14)$$

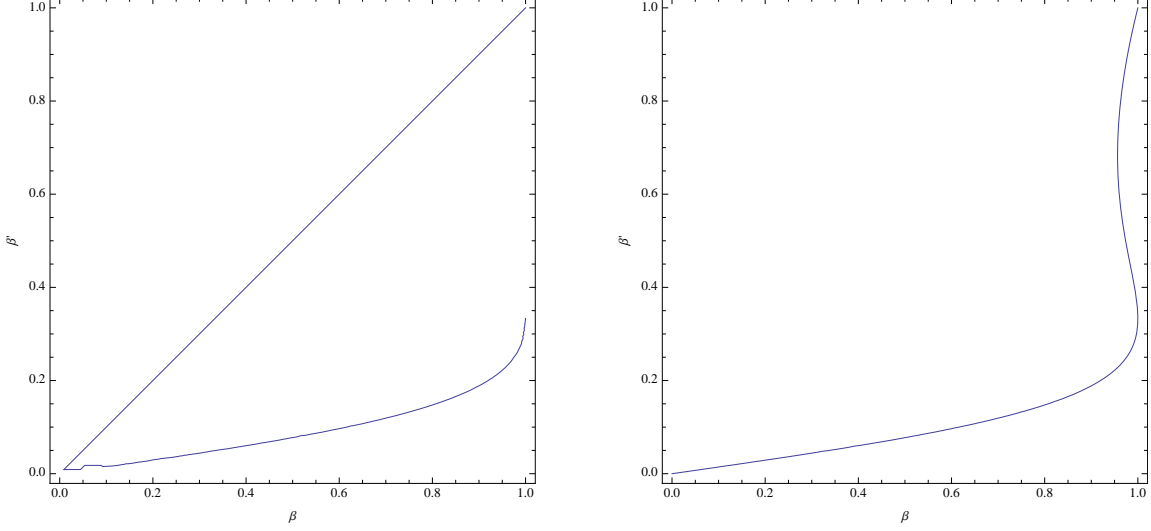


Figure 1: In the left plot, using `ContourPlot`, we plot the solutions to 11 and find both the trivial and nontrivial solutions. To elucidate the interesting behavior in the nontrivial solution, we solve for the nontrivial solution, given in Eq. 12, and plot the solution as a function of β in the right plot.

which gives us

$$7\beta'^2 - 8\beta\beta' + \beta^2 = 0 \quad \rightarrow \quad (\beta - \beta')(\beta - 7\beta') = 0, \quad (15)$$

again taking the nontrivial solution, we find that in the non-relativistic limit $\beta \rightarrow 0$, β' goes as $\beta' = \frac{1}{7}\beta$.

Now we consider the case of an ultra-relativistic shock, $\beta \rightarrow 1$, we see that for our relation in Eq. 11 as $\beta \rightarrow 1$, $\gamma \rightarrow \infty$, and thus we find

$$3\beta'^2 - 4\beta' + 1 = 0 \quad \rightarrow \quad (3\beta' - 1)(\beta' - 1) = 0. \quad (16)$$

so as $\beta \rightarrow 1$, $\beta' \rightarrow \frac{1}{3}$, for the nontrivial solution. Just as before we now want to find the leading order behavior of β' with respect to β . We start by expanding our solution in Eq. 12 around $\beta' = 1/3$

$$\beta = \frac{9\beta'^2 + 7}{15\beta'^2 + 1}\beta' = 1 - \frac{1}{4}(3\beta' - 1)^2 + \mathcal{O}(3\beta' - 1)^3. \quad (17)$$

Rearranging, we find that

$$\beta' = \frac{1}{3} + \frac{2}{3}\sqrt{1 - \beta}, \quad (18)$$

which tells us the behavior of β' in the neighborhood of $\beta = 1/3$ as $\beta \rightarrow 1$.

f) The pre-shock 4-velocity in the rest frame of the shock is $u^\alpha = (\gamma, \gamma\beta, 0, 0)$ and the post-shock 4-velocity in the rest frame of the shock is $u'^\alpha = (\gamma', \gamma'\beta', 0, 0)$. We want to transform into the frame of the pre-shocked gas defined by the unprimed variables. Then we want to find a transformation of the Lorentz factor for the post-shock gas. We want to boost the 4-velocity of the post-shock gas into the frame of the pre-shocked gas, as such

$$u_{\text{ps}}^\alpha = \Lambda^\alpha_\beta u'^\beta \rightarrow \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma' \\ \beta'\gamma' \\ 0 \\ 0 \end{pmatrix} \quad (19)$$

As $u_{\text{ps}}^0 = \gamma_{\text{ps}}$, we find that $\gamma_{\text{ps}} = \gamma\gamma'(1 - \beta\beta')$. In the ultra-relativistic limit, where $\beta \rightarrow 1$, $\beta' \rightarrow \frac{1}{3}$ and thus $\gamma' \rightarrow \frac{3}{2\sqrt{2}}$, we find that $\gamma_{\text{ps}} \rightarrow \frac{1}{\sqrt{2}}\gamma$, which is the desired relation.

2. Electrodynamics based on the action

We begin by considering the action for a particle in an electromagnetic field

$$S = S_{\text{EM}} + S_{\text{particle}} + S_{\text{int}} = -\frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu} - m \int_{\mathcal{P}} d\tau + \int d^4x A_{\mu} J^{\mu} \quad (20)$$

where the three terms correspond to the gauge kinetic term for the electromagnetic field, the particle moving along its world-line, for a path \mathcal{P} , and an interaction term, respectively.

a) First, we want to show that the action is gauge invariant, i.e. the variation in the action $\delta S = 0$, under a gauge transformation, $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\chi$, where χ is some scalar field. We find the variation of the action to be

$$\delta S = -\frac{1}{16\pi} \int d^4x 2F^{\mu\nu} \delta F_{\mu\nu} + \int d^4x \partial_{\mu}\chi J^{\mu}. \quad (21)$$

It should not be surprising that the gauge field strength is gauge invariant. Under a gauge transformation we find

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + \partial_{\mu}\partial_{\nu}\chi - \partial_{\nu}\partial_{\mu}\chi, \quad (22)$$

and thus $\delta F = 0$ because partial derivatives commute. Integrating by parts for the second term in Eq. 21, we find that

$$\delta S = \int d^4x (\partial_{\mu}(\chi J^{\mu}) - \chi \partial_{\mu} J^{\mu}). \quad (23)$$

As total derivatives vanish in the action, we are left with just the second term. Because the variation of the action must vanish for an arbitrary χ , we are left to conclude that $\partial_{\mu} J^{\mu} = 0$, and thus the charge is conserved.

b) We now consider a particle with charge e , we can construct the 4-vector charge for a point particle as

$$J^{\mu} = e \int_{\mathcal{P}} d\lambda \delta^{(4)}(x - x(\lambda)) \frac{dx^{\mu}(\lambda)}{d\lambda}, \quad (24)$$

where λ is some parametrization of the particle's world-line. Inserting this definition into the interaction term we find that

$$S_{\text{int}} = e \int_{\mathcal{P}} d\lambda A_{\mu} \left(\int d^4x \delta^{(4)}(x - x(\lambda)) \frac{dx^{\mu}(\lambda)}{d\lambda} \right) = e \int_{\mathcal{P}} d\lambda A_{\mu}(x(\lambda)) \frac{dx^{\mu}(\lambda)}{d\lambda}. \quad (25)$$

Now we want to reexpress the particle term in the action, S_{part} , in terms of this parametrization. More explicitly, we want to reparametrize the proper time in the integrand using the λ parametrization of the world-line. From $d\tau^2 = -g_{\mu\nu} dx^{\mu} dx^{\nu}$, we know that

$$\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}}. \quad (26)$$

Therefore the particle's action takes the form

$$S_{\text{part}} = -m \int_{\mathcal{P}} d\tau = -m \int_{\mathcal{P}} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (27)$$

So we can rewrite the particle and interaction terms in the action in terms of the λ parametrization of the world-line as

$$S_{\text{int}} + S_{\text{part}} = \int_{\mathcal{P}} d\lambda \left(-m \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} + e A_\mu \frac{dx^\mu}{d\lambda} \right). \quad (28)$$

We now want to consider the variation of the two terms in the action above by varying the functions $x^\mu(\lambda)$, where we take $x^\mu(\lambda) \rightarrow x^\mu(\lambda) + \delta x^\mu(\lambda)$. We consider the variation of the particle world-line first,

$$\delta S_{\text{part}} = \int_{\mathcal{P}} d\lambda \left(-2m \frac{1}{2} \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{-1/2} \left(-g_{\mu\nu} \frac{d(\delta x^\mu)}{d\lambda} \frac{dx^\nu}{d\lambda} \right) \right), \quad (29)$$

where the first factor of two comes from the fact that the variations of $\frac{dx^\mu}{d\lambda}$ and $\frac{dx^\nu}{d\lambda}$ are the same. Proceeding, we find

$$\delta S_{\text{part}} = \int_{\mathcal{P}} d\lambda \left(m g_{\mu\nu} \frac{d\lambda}{d\tau} \frac{d(\delta x^\mu)}{d\lambda} \frac{dx^\nu}{d\lambda} \right) = \int_{\mathcal{P}} d\lambda \left(m \frac{d(\delta x^\mu)}{d\lambda} \frac{dx_\mu}{d\tau} \right) \quad (30)$$

$$= \int_{\mathcal{P}} d\lambda \left(\frac{d}{d\lambda} (m \delta x^\mu u_\mu) - m (\delta x^\mu) \frac{du_\mu}{d\lambda} \right) = \int_{\mathcal{P}} d\lambda \delta x^\mu \left(-m \frac{du_\mu}{d\lambda} \right), \quad (31)$$

where in the last step we used our definition of the 4-velocity and rewrote the expression using a total derivative, which vanishes when evaluated at the ends of the world-line \mathcal{P} . Now we want to vary the interaction term in the action, it should be clear that the variation of the gauge field is

$$\delta A_\mu(x(\lambda)) = \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu(\lambda). \quad (32)$$

Thus, we find that

$$\delta S_{\text{int}} = \int_{\mathcal{P}} d\lambda e \left(\partial_\nu A_\mu \delta x^\nu \frac{dx^\mu}{d\lambda} + A_\mu \frac{d(\delta x^\mu)}{d\lambda} \right) = \int_{\mathcal{P}} d\lambda e \left(\partial_\nu A_\mu \delta x^\nu \frac{dx^\mu}{d\lambda} - \delta x^\mu \frac{dA_\mu}{d\lambda} \right) \quad (33)$$

$$= \int_{\mathcal{P}} d\lambda e \delta x^\mu \left(\partial_\mu A_\nu \frac{dx^\nu}{d\lambda} - \partial_\nu A_\mu \frac{dx^\nu}{d\lambda} \right), \quad (34)$$

where in the first line we rewrote the second term using a total derivative, and in the second line we have switched the dummy indices in the first term and used the chain rule in the second term. Putting the two terms together, we find that

$$\delta S = \delta S_{\text{part}} + \delta S_{\text{int}} = \int_{\mathcal{P}} d\lambda \delta x^\mu \left(-m \frac{du_\mu}{d\lambda} + e \frac{dx^\nu}{d\lambda} F_{\mu\nu} \right). \quad (35)$$

Since the variation in the action must vanish for arbitrary δx^μ , we find the equations of motion for the charged particle

$$e F_{\mu\nu} \frac{dx^\nu}{d\lambda} - m \frac{du_\mu}{d\lambda} = 0. \quad (36)$$

By reparametrization invariance, we can reparametrize the world-line of the particle in terms of the coordinate time t . Evaluating the free index over the spatial components, we find that

$$m \frac{du^i}{dt} = e F_{i0} \frac{dx^0}{dt} + e F_{ij} \frac{dx^j}{dt} = e (E_i + \epsilon_{ijk} v_j B_k), \quad (37)$$

and we have thus found the Lorentz force law

$$\vec{F} = m\vec{a} = e (\vec{E} + \vec{v} \times \vec{B}), \quad (38)$$

which is the equation of motion of a charged particle moving through an electromagnetic field.

c) We now want to consider the variation of the action by varying the gauge field $A_\mu \rightarrow A_\mu + \delta A_\mu$, and in doing so find the equations of motion for the gauge field. Under the variation δA_μ , we find that

$$\delta S = \int d^4x \left(-\frac{1}{16\pi} 2F^{\mu\nu} \delta F_{\mu\nu} + \delta A_\mu J^\mu \right) = \int d^4x \left(-\frac{1}{8\pi} F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + \delta A_\mu J^\mu \right) \quad (39)$$

$$= \int d^4x \left(-\frac{1}{4\pi} F^{\mu\nu} \partial_\mu \delta A_\nu + \delta A_\mu J^\mu \right) = \int d^4x \delta A_\nu \left(\frac{1}{4\pi} \partial_\mu F^{\mu\nu} + J^\nu \right). \quad (40)$$

The first factor of two in the first line comes from the realization that varying each field strength gives the same contribution, and the second factor of two in the second line comes from the fact that the second contraction can be rewritten in the form of the first contraction using the antisymmetry of $F_{\mu\nu}$ and relabeling dummy indices. In the last step we have used a total derivative to pull out the variation. Since the term in the brackets must vanish for an arbitrary variation, we find that the equations of motion for the gauge field are

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu, \quad (41)$$

where we have used the antisymmetry of F and relabeled indices to get the expression into the familiar form. Thus we have recovered the Gauss-Ampère law.