Ph236 Homework 2 Solutions

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1. Basis Independence of Contractions

The tensor **S** is a rank $\binom{2}{1}$ tensor, $S^{\alpha\beta}{}_{\gamma}$, and **T** is a rank $\binom{1}{0}$ tensor, $T^{\beta} = S^{\alpha\beta}{}_{\alpha}$. More explicitly,

$$T(\tilde{k}) = \sum_{\alpha=0}^{3} S(\tilde{\omega}^{\alpha}, \tilde{k}, e_{\alpha}). \tag{1}$$

Let's consider an unprimed basis and a primed basis, where the two bases are related by a basis transformation L. The transformation acts on basis vecotrs and basis one-forms as

$$\vec{e}_{\mu'} = L^{\mu}{}_{\mu'}\,\vec{e}_{\mu} \qquad \text{and} \qquad \tilde{\omega}^{\mu'} = (L^{-1})^{\mu'}{}_{\mu}\,\tilde{\omega}^{\mu}. \tag{2}$$

We now consider our contracted tensor T in the primed basis,

$$T^{\nu'} = S^{\mu'\nu'}{}_{\mu'} = (L^{-1})^{\mu'}{}_{\mu}\,(L^{-1})^{\nu'}{}_{\nu}\,L^{\rho}{}_{\mu'}S^{\mu\nu}{}_{\rho} = \delta^{\rho}{}_{\mu}(L^{-1})^{\nu'}{}_{\nu}\,S^{\mu\nu}{}_{\rho} \tag{3}$$

$$= (L^{-1})^{\nu'}{}_{\nu}\,S^{\mu\nu}{}_{\mu} = (L^{-1})^{\nu'}{}_{\nu}\,T^{\nu}. \tag{4}$$

Since the components of the contracted tensor T^{β}, transform as $T^{\nu'} = (L^{-1})^{\nu'}{}_{\nu}\,T^{\nu}$, just as we would expect any rank $\binom{1}{0}$ to transform, we may conclude that the contraction **T** of **S** is independent of the choice of basis.

2. Forms in 2 Dimensions

a) We are working in two-dimensional Euclidean space $\mathbb{R}^2$. First we want to find the Hodge dual of a scalar f,

$$\star f = (\star f)^{ij} = \epsilon^{ij} f. \tag{5}$$

Now we want to find the dual of a vector,

$$(\star v)^{j} = \epsilon^{ij} v_i. \tag{6}$$

Explicitly, we see that for a vector $v = (v^1, v^2)$, the dual vector is $\star v = (-v^2, v^1)$. The geometric interpretation of this is that taking the dual of a vector corresponds to a $\pi/2$ rotation counterclockwise. Thus we may preempt the next question and guess that taking the dual of the dual vector will correspond to a rotation by π, and gives us $-\vec{v}$.

$$(\star \star v)^j = \epsilon^{ij}(\star v_i) = \epsilon^{ij}\epsilon_{ki}v^k = -\delta^j{}_k v^k = -v^k, \tag{7}$$

as expected. Now the dual of the dual of a scalar,

$$(\star \star f) = \star(\epsilon^{ij} f) = \frac{1}{2}\epsilon_{ij}\epsilon^{ij} f = \frac{1}{2}2f = f. \tag{8}$$

A technical aside
For fun let's try and rederive the same results in a more general discussion of some of these concepts. First let's review some general definitions which will be useful in our discussion, we write an r-form in a coordinate basis as

$$\omega = \frac{1}{r!}\omega_{\mu_1 \ldots \mu_r}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \tag{9}$$

where the space of r-forms on an m-dimensional manifold $\mathcal{M}$ as is $\Omega^r(\mathcal{M})$. Dropping the basis allows us to write $\omega = \omega_{\mu_1 \ldots \mu_r}\, dx^{\mu_1}$, where the prefactor is absorbed in the antisymmetrization. The exterior derivative on an r-form is defined as the map $d : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$. Thus for a general r-form we find the exterior derivative to be

$$d\omega = \frac{1}{r!}\partial_\alpha \omega_{\mu_1 \ldots \mu_r}\, dx^\alpha \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}. \tag{10}$$

Dropping the coordinate basis allows us to recover the familiar $d\omega = (r+1)\partial_\alpha \omega_{\mu_1 \ldots \mu_r}$, where r is the rank of the form. The $r+1$ prefactor comes from the antisymmetrization of all $r+1$ indices in the basis. Since $\dim \Omega^r(\mathcal{M}) = \dim \Omega^{m-r}(\mathcal{M})$ on an m-dimensional manifold $\mathcal{M}$, it is natural to define the Hodge dual action as the map $\star : \Omega^r(\mathcal{M}) \to \Omega^{m-r}(\mathcal{M})$. Thus for a general r-form we find the Hodge dual to be

$$\star\, \omega = \frac{\sqrt{|g|}}{r!(m-r)!}\omega_{\mu_1 \ldots \mu_r}\epsilon^{\mu_1 \ldots \mu_r}{}_{\mu_{r+1} \ldots \mu_m}\, dx^{\mu_{r+1}} \wedge \ldots \wedge dx^{\mu_m}. \tag{11}$$

Applying this operation twice we find that and using the identity,

$$\epsilon^{\alpha_1 \ldots \alpha_r \alpha_{r+1} \ldots \alpha_m}\epsilon_{\alpha_1 \ldots \alpha_r \beta_{r+1} \ldots \beta_m} = r!(m-r)!\,(\det g)^{-1}\, \delta^{[\alpha_{r+1}}{}_{\beta_{r+1}}\delta^{\alpha_{r+2}}{}_{\beta_{r+2}} \ldots \delta^{\alpha_m]}{}_{\beta_m} \tag{12}$$

For an r-form $\omega \in \Omega^r(\mathcal{M})$, we find that for Euclidean metric signature $(0, m)$ and for Lorentzian metric signature $(1, m-1)$, that

$$\begin{aligned}(0, m) :& \quad \star \star \omega = +(-1)^{r(m-r)}\omega \\ (1, m-1) :& \quad \star \star \omega = -(-1)^{r(m-r)}\omega\end{aligned}$$

Now we focus our general discussion to $\mathbb{R}^2$, where the signature is $(0, 2)$. Now for an r-form $\omega \in \Omega^r(\mathbb{R}^2)$, the expression for $\star \star \omega$ becomes

$$\star \star \omega = +(-1)^{r(2-r)}\omega. \tag{13}$$

Formally, thinking about a scalar f in the language of forms means we think about it as a 0-form, which means we must make the definition $f \in \Omega^0 = \mathbb{R}$. Proceeding we see that

$$\star\star f = +f, \tag{14}$$

whereas for a 1-form $v \in \Omega^1(\mathbb{R}^2)$, we find that

$$\star\star v = -v. \tag{15}$$

b) We now consider a vector field $v(x)$, or equivalently a 1-form field ω. The two ways we may construct scalar fields are $f = \star d\omega$ and $h = \star d \star \omega$. We should see that these are both scalars as $d\omega$ is a 2-form and $\star d\omega$ is a $(2-2=0)$-form. Similarly, $\star d \star \omega$ is a $(2-(1+2-1)=0)$-form. Let's evaluate these explicitly, we start with

$$(d\omega)_{ij} = 2\partial_{[i}\omega_{j]} = \partial_i\omega_j - \partial_j\omega_i. \tag{16}$$

This inherent antisymmetry is clear from the graded commumtivity in the coordinate basis of wedge products,

$$d\omega = \partial_\mu\omega_\nu dx^\mu \wedge dx^\nu \tag{17}$$

Continuing,

$$\star\, d\omega = \frac{1}{2}\epsilon^{ij}(\partial_i\omega_j - \partial_j\omega_i) = \epsilon^{ij}\partial_i\omega_j \tag{18}$$

where we have used the antisymmetry of ϵ. We can identitfy this as the curl of ω. We recall that in three dimensions the curl of a one-form (or equivalently, a vector) gives us a vector, but in two dimensions the curl of a vector gives a scalar as we see above. Now we consider the second way to construct a scalar $\star d \star \omega$,

$$\begin{aligned} \star d \star \omega &= \star d(\epsilon^i{}_j\, \omega_i\, dx^j) = \star d(g^{ik}\epsilon_{kj}\omega_i\, dx^j) \\ &= 2 \star (g^{ik}\epsilon_{kj}\, \partial_\ell\omega_i\, dx^j \wedge dx^\ell) \\ &= \epsilon^{j\ell}g^{ik}\epsilon_{kj}\, \partial_\ell\omega_i = \delta^\ell{}_k g^{ik}\, \partial_\ell\omega_i = g^{i\ell}\, \partial_\ell\omega_i = \partial_\ell\omega^i. \end{aligned}$$

We identify this as the divergence of ω. We also note that since our space is Euclidean and obviously has a well-defined metric, we have not concerned ourselves in the above discussion with the distinction between one-forms and vectors.

3. Existence of 1-form Potentials

a) Consider a closed 2-form H, where the last row vanishes $H_{ni} = 0$ for all values of i, and $H_{ij}(x^1, \ldots, x^{n-1}, 0) = 0$, i.e. $H_ij = 0$ on $\mathbb{R}^{n-1}$. We start with the fact that H is closed,

$$(dH)_{ijk} = 3\partial_{[i}H_{jk]} = \frac{3}{3!}(\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij} - \partial_i H_{kj} - \partial_j H_{ik} - \partial_k H_{ji}) = 0 \tag{19}$$

where $1 \geq i, j, k \leq n$. Using the antisymmetry of H we find that

$$(dH)_{ijk} = \partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij} = 0. \tag{20}$$

We note that as the entire object is antisymmetric in all three indices there can be not repeated index, i.e. $i \neq j \neq k$. We split the above equation into components and take k to run over just n, and i, j to run over the other coordinates, $1 \geq i, j < n$. We find that

$$\partial_i H_{jn} + \partial_j H_{ni} + \partial_n H_{ij} = 0. \tag{21}$$

The first two terms vanish as $H_{ni} = 0$ on all of $\mathbb{R}^{\ltimes}$, which leaves us with the statement that $\partial_n H_{ij} = 0$. This means that $H = H(x^1, \ldots, x^{n-1})$ and is only a function of coordinates on the $\mathbb{R}^{n-1}$ subspace. Since we already know that H_{ij} on $\mathbb{R}^{n-1}$ we may conclude that $H = 0$ everywhere on $\mathbb{R}^n$.

b) We now take the dimension of our space to be $n = 2$ and claim that there exists a two-form F that is closed. In two dimensions any two-form is automatically closed. In case this isn't immediately obvious consider taking the exterior derivative of a two-form in a coordinate basis of forms, $d\omega = \partial_i \omega_{jk} dx^i \wedge dx^j \wedge dx^k$, where $i \neq j \neq k$. Thus as the indices only run over two dimensions, $d\omega = 0$ by the antisymmetry of the indices. Now we define a one-form $A = A_i dx^i$ and take the exterior derivative

$$dA = (\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j. \tag{22}$$

In two dimensions F one has one degree of freedom, $F_{12} = -F_{21}$, as $F_{11} = F_{22} = 0$. Let that one degree of freedom be given by some function $f(x^1, x^2)$, i.e. $F_{12} = -F_{21} = f(x^1, x^2)$. Then as dA is a two-form, we may write

$$f(x^1, x^2) = \partial_1 A_2 - \partial_2 A_1. \tag{23}$$

There exist many choices of $A^1(x^1, x^2)$ and $A^2(x^1, x^2)$ that satisfy our conditions for F. But we may make the choice that $A^1(x^1, x^2) = 0$ and

$$A^2(x^1, x^2) = \int_0^{x^1} f(y^1, x^2) dy^1. \tag{24}$$

Thus for any 2-form F in two dimensions, there exists a 1-form A, where we may write $F = dA$.

c) We now want to extend our argument in part b, to $n > 2$ dimensions. All we need to do is show that if in $d = n - 1$ there exists a 2-form F, such that $F = dA$, then this also holds for $d = n$. Let $F = F(x^1, \ldots, x^n)$ be a closed 2-form in n dimensions and let there be another closed 2-form in $n - 1$ dimensions $G(x^1, \ldots, x^{n-1}) = F(x^1, \ldots, x^{n-1}, 0)$, where $G = G_{ij}$ for $1 \geq i, j \leq n - 1$. From our argument in part b, there then exists a 1-form field $\sigma_i(x^1, \ldots, x^{n-1})$ defined on $\mathbb{R}^{n-1}$, the $n-1$ dimensional plane, so $G = d\sigma$ on $x^n = 0$. Thus we may define

$$A_i(x^1, \ldots, x^n) = \sigma_i(x^1, \ldots, x^{n-1}) + \int_0^{x^n} F_{ni}(x^1, \ldots, y^n) dy^n \tag{25}$$

where the index i runs over $i = 1, \ldots, n - 1$. Just as in part b, we fix the degrees of freedom of A and take $A_n = 0$. Now consider a 2-form H defined as $H = F - dA$, which is clearly a closed form. We now want to show that all the components of H vanish. Clearly by antisymmetry H_{nn} vanishes. Now consider H_{in} where $1 \geq i \leq n - 1$,

$$H_{in} = F_{in} - \partial_i A_n + \partial_n A_i = F_{in} + \partial_n \sigma_i - \frac{\partial}{\partial x^n} \left(\int_0^{x^n} F_{in}(x^1, \ldots, y^n) dy^n \right) = F_{in} - F_{in} = 0 \tag{26}$$

using the definition of A above and the fact that σ vanishes in the x^n direction. Now we want to consider H_{ij},

$$\begin{aligned} H_{ij} &= F_{ij} - \partial_i A_j + \partial_j A_i && (27) \\ &= F_{ij} - (d\sigma)_{ij} - \frac{\partial}{\partial x^i} \int_0^{x^n} F_{nj}(x^1, \ldots, y^n) dy^n + \frac{\partial}{\partial x^j} \int_0^{x^n} F_{ni}(x^1, \ldots, y^n) dy^n && (28) \\ &= F_{ij} - (d\sigma)_{ij} - \int_0^{x^n} (\partial_i F_{nj} - \partial_j F_{ni}) dy^n && (29) \\ &. && (30) \end{aligned}$$

Since $1 \geq i, j \leq n-1$, the integral will clearly vanish in the integration over the n-th direction and since we defined $F_{ij} = (d\sigma)_{ij}$, we find that $H_{ij} = 0$. We have found that H aligns with the statement in part a, that the last row vanishes $H_{in} = 0$ on $\mathbb{R}^{\ltimes}$, and $H_{ij} = 0$ on the $\mathbb{R}^{n-1}$ subspace, and thus H vanishes everywhere. We have then found that everywhere in $\mathbb{R}^{\ltimes}$ F can be written as $F = dA$, meaning we have explicitly constructed the 2-form field F as the exterior derivative of a 1-form A in n dimensions.

This result is useful in differential geometry and is often discussed in a more general context. The more general statement of this goes by the name Poincaré lemma and states that for any contractible open set on a manifold, or as is more applicable in our case, any contractible domain in $\mathbb{R}^n$, $\omega \in \Omega^r$ such that $d\omega = 0$, for $r > 0$ there exists $\alpha \in \Omega^{r-1}$ such that $\omega = d\alpha$. An r-form that may be written as $\omega = d\alpha$, where $\alpha \in \Omega^{r-1}$, is called an exact form. In other words, for any contractible domain in $\mathbb{R}^n$ a closed r-form is also locally an exact form. The measure of which closed r-forms are exact forms on some manifold $\mathcal{M}$ is called the r-th de Rham cohomology $H^r(\mathcal{M})$. Thus if all closed 2-forms are exact forms on $\mathbb{R}^2$ then the second de Rham cohomology class is trivial, $H^2(\mathbb{R}^2) = 0$.

d) From the nilpotency of the exterior derivative $d^2\omega = 0$, we see that for an exact 2-form $F = dA$, the 1-form can be rewritten as $A + d\alpha$, where $\alpha \in \Omega^0$ (i.e. a scalar), without changing our definition of F. Thus the 1-form potential is not unique. This is equivalent to saying that the 1-form A is only unique up to an exact 1-form $\beta = d\alpha$. This is just a mathematical restatement of the gauge invariance of $F_{\mu\nu}$, which is the field strength of the gauge field A_μ, where a general gauge transformation is given by $A_\mu \to A_\mu + \partial_\mu f$.

4. Electromagentic Potential

We know that F is an exact 2-form, given in terms of a 1-form A as $F = dA$. In the language of forms in a coordinate basis, where a general r-form $\omega \in \Omega^r$ can be written as $\omega = \omega_{\mu_1 \ldots \mu_r} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$, the field strength F and potential A are written as

$$A = A_\mu dx^\mu \qquad F = F_{\mu\nu} \, dx^\mu \wedge dx^\nu. \tag{31}$$

Thus we have

$$dA = 2\partial_{[\mu} A_{\nu]} \, dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu) \, dx^\mu \wedge dx^\nu \tag{32}$$

using the graded commutivity of the wedge product (in which the antisymmetry of forms in encoded). Thus we arrive at the familiar expression

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{33}$$

Splitting the potential A into its time component Φ and spatial components, $A_\mu = (-\Phi, A_i)$ and recalling that the electromagnetic field strength encodes the electric and magnetic fields as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{34}$$

or equivalently, $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ijk} B_k$. Thus, we find that

$$F_{i0} = \partial_i A_0 - \partial_0 A_i = E_i \qquad \rightarrow \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \tag{35}$$

and that

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k \tag{36}$$

contracting with ϵ_{ijm} we find that

$$B_m = \epsilon_{ijm} \partial_i A_j \qquad \rightarrow \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{37}$$