Ph236 Homework 1 Solutions

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Problem 1: Superluminal Motion

a) Let (x^0, x^1, x^2, x^3) be the coordinates of the event where the light is emitted. Later, the light is observed at time t by an observer located a distance D away in the $+1$ direction. This event has coordinates $(t, D, 0, 0)$, where $D \gg x^i$, since the observer is "far away." The separation vector $\vec{\xi}$ of these two events is $\vec{\xi} = (t - x^0, D - x^1, -x^2, -x^3)$. Since light travels with the speed of light, $\vec{\xi}$ is light-like, and so we get

$$0 = |\vec{\xi}|^2 = -(t - x^0)^2 + (D - x^1)^2 + (x^2)^2 + (x^3)^2$$
$$\Leftrightarrow \ t = x^0 + \sqrt{(D - x^1)^2 + (x^2)^2 + (x^3)^2} \approx x^0 - x^1 + D. \tag{1}$$

We only used the positive square root because causality dictates that $t \geq x^0$, and we neglected the $(x^2)^2$ and $(x^3)^2$ terms int he square root, since they are both much smaller than $(D - x^1)^2$.

b) If we express the given change of basis as

$$\vec{e}_{\alpha'} = L^{\beta}{}_{\alpha'} \vec{e}_{\beta}, \tag{2}$$

then the components transform as

$$x^{\alpha'} = (L^{-1})_{\beta}{}^{\alpha'} x^{\beta}. \tag{3}$$

Thus we immediately get

$$|(L^{-1})_{\beta}{}^{\alpha'}| = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

and so

$$|L^{\beta}{}_{\alpha'}| = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Now using (2) we find

$$\begin{aligned} g_{\alpha'\beta'} &= \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'} = (L^{\mu}{}_{\alpha'}\vec{e}_{\mu}) \cdot (L^{\nu}{}_{\beta'}\vec{e}_{\nu}) = L^{\mu}{}_{\alpha'}L^{\nu}{}_{\beta'}(\vec{e}_{\mu} \cdot \vec{e}_{\nu}) \\ &= L^{\mu}{}_{\alpha'}L^{\nu}{}_{\beta'}g_{\mu\nu}. \end{aligned} \tag{6}$$

Since we are in Minkowski space, we have $g_{\mu\nu} = \eta_{\mu\nu}$, which is non-zero only when $\mu = \nu$. We thus find and the other components of $g_{\alpha'\beta'}$ are clearly 0. So we found

$$|g_{\alpha'\beta'}| = \begin{bmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

c) Recall that $V_{\perp}$ is the observed transverse velocity. Since the observer is located far away in the x^1 direction, motion in the x^1 direction cannot be observed, so only motion in the x^2 direction is observed. Furthermore, we have shown in part (a) that the light emitted by the blob is observed at time $x^0 - x^1 = x^{0'}$, which means that the observed transverse velocity is

$$V_{\perp} = \frac{dx^2}{dx^{0'}} = \frac{dx^{2'}}{dx^{0'}}, \tag{8}$$

since $x^2 = x^{2'}$.

d) Recall that the 4-velocity in the primed coordinate system is given by

$$u^{\alpha'} = \frac{dx^{\alpha'}}{d\tau} = \frac{dx^{\alpha'}}{dx^{0'}}\frac{dx^{0'}}{d\tau} = \frac{dx^{0'}}{d\tau}\left(1, \frac{dx^{1'}}{dx^{0'}}, \frac{dx^{2'}}{dx^{0'}}, 0\right), \tag{9}$$

since $x^{\alpha'}$ is parametrized in terms of $x^{0'}$ and since $x^3 = x^{3'} = 0$, and where τ is the proper time. The 4-velocity is timelike if its norm is negative, hence we need $u^{\alpha'}u_{\alpha'} < 0$. This gives

$$\begin{aligned} 0 > u^{\alpha'}u_{\alpha'} &= g_{\alpha'\beta'}u^{\alpha'}u^{\beta'} \\ &= \left(\frac{dx^{0'}}{d\tau}\right)^2 \left(-\frac{1}{2}(1)\frac{dx^{1'}}{dx^{0'}} - \frac{1}{2}(1)\frac{dx^{1'}}{dx^{0'}} + \left(\frac{dx^{2'}}{dx^{0'}}\right)^2\right) \\ \Leftrightarrow \frac{dx^{1'}}{dx^{0'}} &> \left(\frac{dx^{2'}}{dx^{0'}}\right)^2 = V_{\perp}^2. \end{aligned} \tag{10}$$

Note that the above does not include any constants (e.g. 1), and so $dx^{1'}/dx^{0'}$ and $V_\perp^2$ can be arbitrarily large. Thus for any non-negative value of $V_\perp$, we simply take (for example) $dx^{1'}/dx^{0'} = 2V_\perp^2$ and then the 4-velocity of the blob is timelike.

e) As shown in (9), the 4-velocity is

$$u^{\alpha'} = \frac{dx^{0'}}{d\tau}\left(1, \frac{dx^{1'}}{dx^{0'}}, \frac{dx^{2'}}{dx^{0'}}, 0\right) = \frac{dx^{0'}}{d\tau}\left(1, q, V_\perp, 0\right), \tag{11}$$

and from class we know that

$$\frac{d\tau}{dx^{0'}} = \sqrt{-g_{\alpha'\beta'}\frac{dx^{\alpha'}}{dx^{0'}}\frac{dx^{\beta'}}{dx^{0'}}} = \sqrt{\frac{1}{2}(1)(q) + \frac{1}{2}(1)(q) - V_\perp^2} = \sqrt{q - V_\perp^2}. \tag{12}$$

Thus the 4-velocity components $u^{\alpha'}$ are

$$u^{\alpha'} = \frac{1}{\sqrt{q - V_\perp^2}}(1, q, V_\perp, 0). \tag{13}$$

f) The components of the 4-velocity transform in the same way as the components of a vector and so we get

$$u^{\alpha} = \left(\frac{u^{0'} + u^{1'}}{2}, \frac{u^{1'} - u^{0'}}{2}, u^{2'}, u^{3'}\right) = \frac{1}{\sqrt{q - V_\perp^2}}\left(\frac{1+q}{2}, \frac{q-1}{2}, V_\perp, 0\right). \tag{14}$$

Now the Lorentz factor γ is simply u^0, hence it is

$$\gamma = u^0 = \frac{1+q}{2\sqrt{q - V_\perp^2}}, \tag{15}$$

which is exactly what we need to show. □

Recall that the condition from part (d) is $q > V_\perp^2$, and so the square root is always defined. To find the minimum possible value of γ we set $d\gamma/dq = 0$ and we find that the only solution to that is $q = 1 + 2V_\perp^2$. Evaluating $d^2\gamma/dq^2$ at $q = 1 + 2V_\perp^2$ yields $\left(4(1 + V_\perp^2)^{3/2}\right)^{-1} > 0$ and so $q = 1 + 2V_\perp^2$ is indeed the minimum of γ. Also note that $q > V_\perp^2$ is satisfied. Now the question remains for what value of $V_\perp$ the minimum value of γ occurs. Substituting $q = 1 + 2V_\perp^2$ into the equation for γ gives $\gamma = \sqrt{1 + V_\perp^2}$, which is clearly minimized at $V_\perp = 1$, since we require $V_\perp \geq 1$. So the minimum value of γ occurs at $V_\perp = 1$ and $q = 3$, which satisfies $q > V_\perp^2$ and we have

$$\gamma_{\min} = \frac{1+3}{2\sqrt{3-1}} = \frac{4}{2\sqrt{2}} = \sqrt{2}, \tag{16}$$

as we needed to show. □

Finally, recall that

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \tag{17}$$

where v is the magnitude of the 3-velocity of the blob in the lab frame, and so

$$v = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \tag{18}$$

for $\gamma = \sqrt{2}$, and thus we have shown that $\gamma = \sqrt{2}$ corresponds to a lab frame velocity of $1/\sqrt{2}$ for the blob. $\square$