Lecture XII: Radiation reaction and binary evolution

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I. OVERVIEW

We are now ready to consider the net loss of energy and angular momentum by a system emitting gravitational waves. We first work out the governing formulae, and then apply them to the special case of a binary system.

The recommended reading for this lecture is:

- MTW §36.1–36.8.

II. ENERGY AND ANGULAR MOMENTUM LOSS DUE TO GRAVITATIONAL WAVE EMISSION

A. Preliminary considerations

Recall from last time that the effective stress-energy tensor for gravitational waves was

$$\langle t^{\mu\nu}\rangle = \frac{1}{32\pi}\langle h^{\mathrm{TT}ij,\mu} h^{\mathrm{TT},\nu}_{ij}\rangle. \tag{1}$$

In Lecture XII, we found that the transverse-traceless gauge gravitational wave emitted from a source was

$$h^{\mathrm{TT}}_{ij} = \frac{1}{R}\left(2\ddot Q_{ij} + n_k n_l \ddot Q_{kl}\delta_{ij} + n_i n_j n_k n_l \ddot Q_{kl} - 2n_j n_k \ddot Q_{ik} - 2n_i n_k \ddot Q_{jk}\right), \tag{2}$$

where the quadrupole moment is evaluated at the retarded time $t-R$, and R and n^i are the magnitude and 3-direction of x^i. This can be simplified if we define the spherical projection tensor

$$\Pi_{ij} = \delta_{ij} - n_i n_j, \tag{3}$$

which takes a 3-vector and finds the part perpendicular to $\boldsymbol{n}$. Note that the projection tensor squared is itself, in the sense of $\Pi_{ik}\Pi^k{}_j = \Pi_{ij}$. We then find

$$h^{\mathrm{TT}}_{ij} = \frac{2}{R}\left(\Pi_{ik}\Pi_{jl}\ddot Q_{kl} + \frac{1}{2}\Pi_{ij} n_k n_l \ddot Q_{kl}\right) = \frac{2}{R}\left(\Pi_{ik}\Pi_{jl}\ddot Q_{kl} - \frac{1}{2}\Pi_{ij}\Pi_{kl}\ddot Q_{kl}\right), \tag{4}$$

where in the second equality we used the fact that $\mathbf{Q}$ is traceless. We also know that due to the retarded evaluation of $\mathbf{Q}$ – i.e. evaluated at $t-R$ – we have

$$Q_{ij,k} = -\dot Q_{ij} R_{,k} = -\dot Q_{ij} n_k, \tag{5}$$

with errors of order λ/R where λ is the wavelength of the gravitational waves. This is useful for converting radial to time derivatives.

It will sometimes be profitable below to write Eq. (4) in the matrix form,

$$\mathbf{h}^{\mathrm{TT}} = \frac{2}{R}\left[\boldsymbol{\Pi}\boldsymbol{Q}\boldsymbol{\Pi} - \frac{1}{2}\boldsymbol{\Pi}\,\mathrm{Tr}(\boldsymbol{\Pi}\boldsymbol{Q})\right]. \tag{6}$$

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B. Energy loss

For the energy loss due to GW emission, we want to compute the outgoing flux of energy through a sphere of some radius R:

$$\begin{aligned}
\mathcal{P} &= \frac{1}{32\pi}\int \langle t^{0i}\rangle n_i\, d^2S \\
&= \frac{1}{32\pi}R^2\int \langle t^{0i}\rangle n_i\, d^2n \\
&= -\frac{1}{32\pi}R^2\int \langle \dot{h}^{\rm TT}_{kl} h^{{\rm TT}\,kl,i} n_i\rangle\, d^2n \\
&= \frac{1}{32\pi}R^2\int \langle \dot{h}^{\rm TT}_{kl} \dot{h}^{{\rm TT}\,kl}\rangle\, d^2n \\
&= \frac{1}{32\pi}R^2\int \langle {\rm Tr}(\dot{\mathbf{h}}^{{\rm TT}\,2})\rangle\, d^2n \\
&= \frac{1}{8\pi}\int \left\langle {\rm Tr}\left[\mathbf{\Pi}\dddot{\boldsymbol{Q}}\mathbf{\Pi} - \frac{1}{2}\mathbf{\Pi}{\rm Tr}(\mathbf{\Pi}\dddot{\boldsymbol{Q}})\right]^2\right\rangle d^2n \\
&= \frac{1}{8\pi}\int \left\langle {\rm Tr}(\mathbf{\Pi}\dddot{\boldsymbol{Q}}\mathbf{\Pi})^2 - {\rm Tr}(\mathbf{\Pi}^2\dddot{\boldsymbol{Q}}\mathbf{\Pi}){\rm Tr}(\mathbf{\Pi}\dddot{\boldsymbol{Q}}) + \frac{1}{4}({\rm Tr}\mathbf{\Pi}^2)[{\rm Tr}(\mathbf{\Pi}\dddot{\boldsymbol{Q}})]^2\right\rangle d^2n.
\end{aligned} \tag{7}$$

Recalling that $\mathbf{\Pi}^2 = \mathbf{\Pi}$ and ${\rm Tr}\mathbf{\Pi} = 2$, we may simplify this to

$$\mathcal{P} = \frac{1}{8\pi}\int \left\langle {\rm Tr}(\mathbf{\Pi}\dddot{\boldsymbol{Q}}\mathbf{\Pi}\dddot{\boldsymbol{Q}}) - \frac{1}{2}[{\rm Tr}(\dddot{\boldsymbol{Q}}\mathbf{\Pi})]^2\right\rangle d^2n = \frac{1}{8\pi}\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle \int \left(\Pi_{ik}\Pi_{jl} - \frac{1}{2}\Pi_{ij}\Pi_{kl}\right) d^2n. \tag{8}$$

We now ned to know the integrals over the unit sphere of $\Pi_{ij}\Pi_{kl}$: if we take an angle-average we get

$$\begin{aligned}
&\langle \Pi_{ij}\Pi_{kl}\rangle_{\boldsymbol{n}} \\
&\qquad = \langle \delta_{ij}\delta_{kl} - n_in_j\delta_{kl} - \delta_{ij}n_kn_l + n_in_jn_kn_l\rangle_{\boldsymbol{n}} \\
&\qquad = \delta_{ij}\delta_{kl} - \frac{1}{3}\delta_{ij}\delta_{kl} - \frac{1}{3}\delta_{ij}\delta_{kl} + \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \\
&\qquad = \frac{2}{5}\delta_{ij}\delta_{kl} + \frac{2}{15}\delta_{l(i}\delta_{j)k}.
\end{aligned} \tag{9}$$

Then Eq. (8) becomes

$$\mathcal{P} = \frac{1}{2}\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle\left(\frac{2}{5}\delta_{ik}\delta_{jl} + \frac{2}{15}\delta_{l(i}\delta_{k)j} - \frac{1}{5}\delta_{ij}\delta_{kl} - \frac{1}{15}\delta_{l(i}\delta_{j)k}\right). \tag{10}$$

This simplifies to

$$\mathcal{P} = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle. \tag{11}$$

This is a remarkably simple formula for the energy carried off by gravitational waves. It depends on the third derivative of the quadrupole moment, and in this sense is similar to the electric dipole radiation formula, $\frac{1}{3}\langle \ddot{d}_i\ddot{d}_i\rangle$. In the case of gravitational waves however the leading term is the quadrupole, hence the additional derivative.

C. Angular momentum carried

At a finite distance from the source, the wavefronts are not exactly spherical. Therefore, the effective stress-energy tensor has nonradial components and there is a net "effective angular momentum density" $\epsilon_{ijk}Rn^jt^{0k}$, where the non-radial part of t^{0k} scales as $\propto R^{-3}$. To be specific, the angular momentum carried through a shell is

$$\mathcal{H}_c = \epsilon_{cab}\int x^a\langle t^{bl}\rangle n_l\, d^2S$$

$$\begin{aligned}
&= R^3\epsilon_{cab}\int n^a\langle t^{bl}\rangle n_l\, d^2n \\
&= \frac{1}{32\pi}R^3\epsilon_{cab}\int n^a\langle h^{\mathrm{TT}ij,b}h^{\mathrm{TT}}{}_{ij}{}^{,l}\rangle n_l\, d^2n \\
&= -\frac{1}{32\pi}R^3\epsilon_{cab}\int n^a\langle h^{\mathrm{TT}ij,b}\dot{h}^{\mathrm{TT}}{}_{ij}\rangle\, d^2n \qquad (12)
\end{aligned}$$

Using the rule that $n_{i,j} = \Pi_{ij}/R$, the projection tensor has gradient

$$\Pi_{ij,a} = -\frac{2}{R}\Pi_{a(i}n_{j)}, \qquad (13)$$

and so we find that

$$n_{[a}\partial_{b]}\Pi_{ij} = -\frac{2}{R}n_{[a}\Pi_{b](i}n_{j)} = -\frac{2}{R}n_{[a}\delta_{b](i}n_{j)}, \qquad (14)$$

or

$$\mathcal{L}_c\Pi_{ij} \equiv R\epsilon_c{}^{ab}n_a\partial_b\Pi_{ij} = -2\epsilon_c{}^{ab}n_a\delta_{b(i}n_{j)} = -2\epsilon_{ca(i}n_{j)}n^a, \qquad (15)$$

where we have defined the operator $\mathcal{L}_c \equiv R\epsilon_c{}^{ab}n_a\partial_b$. This operator has a single derivative and thus obeys the product rule. We then find

$$\mathcal{H}_c = -\frac{1}{32\pi}R^2\int\langle\mathrm{Tr}[(\mathcal{L}_c\mathbf{h}^{\mathrm{TT}})\dot{\mathbf{h}}^{\mathrm{TT}}]\rangle\, d^2n. \qquad (16)$$

The operation $\mathcal{L}_c\mathbf{h}^{\mathrm{TT}}$ can be expanded as

$$\begin{aligned}
\mathcal{L}_c\mathbf{h}^{\mathrm{TT}} &= \frac{2}{R}\mathcal{L}_c\left[\mathbf{\Pi}\ddot{\boldsymbol{Q}}\mathbf{\Pi} - \frac{1}{2}\mathbf{\Pi}\mathrm{Tr}(\mathbf{\Pi}\ddot{\boldsymbol{Q}})\right] \\
&= \frac{2}{R}\left[(\mathcal{L}_c\mathbf{\Pi})\ddot{\mathbf{Q}}\mathbf{\Pi} + \mathbf{\Pi}\ddot{\mathbf{Q}}\mathcal{L}_c\mathbf{\Pi} - \frac{1}{2}(\mathcal{L}_c\mathbf{\Pi})\mathrm{Tr}(\mathbf{\Pi}\ddot{\boldsymbol{Q}}) - \frac{1}{2}\mathbf{\Pi}\mathrm{Tr}(\ddot{\mathbf{Q}}\mathcal{L}_c\mathbf{\Pi})\right], \qquad (17)
\end{aligned}$$

and so

$$\mathcal{H}_c = -\frac{1}{16\pi}R\int\mathrm{Tr}\left\langle(\mathcal{L}_c\mathbf{\Pi})\ddot{\mathbf{Q}}\mathbf{\Pi}\dot{\mathbf{h}}^{\mathrm{TT}} + \mathbf{\Pi}\ddot{\mathbf{Q}}(\mathcal{L}_c\mathbf{\Pi})\dot{\mathbf{h}}^{\mathrm{TT}} - \frac{1}{2}(\mathcal{L}_c\mathbf{\Pi})\mathrm{Tr}(\mathbf{\Pi}\ddot{\boldsymbol{Q}})\dot{\mathbf{h}}^{\mathrm{TT}} - \frac{1}{2}\mathbf{\Pi}\mathrm{Tr}(\ddot{\mathbf{Q}}\mathcal{L}_c\mathbf{\Pi})\dot{\mathbf{h}}^{\mathrm{TT}}\right\rangle d^2n. \qquad (18)$$

This result may be simplified as follows:

- Using the fact that $\dot{\mathbf{h}}^{\mathrm{TT}}$ is already transverse (i.e. orthogonal to $\mathbf{n}$), we have $\mathbf{\Pi}\dot{\mathbf{h}}^{\mathrm{TT}} = \dot{\mathbf{h}}^{\mathrm{TT}}$. Then since $\mathbf{h}^{\mathrm{TT}}$ is traceless, the 4th term in Eq. (18) vanishes.
- The first two terms of Eq. (18) are equal using the cyclic property of the trace and the symmetry of the various factors.

This reduces Eq. (18) to

$$\mathcal{H}_c = \frac{1}{8\pi}R\int\left\langle -\mathrm{Tr}[(\mathcal{L}_c\mathbf{\Pi})\ddot{\mathbf{Q}}\dot{\mathbf{h}}^{\mathrm{TT}}] + \frac{1}{4}\mathrm{Tr}(\mathbf{\Pi}\ddot{\mathbf{Q}})\mathrm{Tr}(\dot{\mathbf{h}}^{\mathrm{TT}}\mathcal{L}_c\mathbf{\Pi})\right\rangle d^2n. \qquad (19)$$

A final simplification is achieved by recalling the formula for $\mathcal{L}_c\mathbf{\Pi}$: from Eq. (15),

$$\mathrm{Tr}(\dot{\mathbf{h}}^{\mathrm{TT}}\mathcal{L}_c\mathbf{\Pi}) = -2\dot{h}^{\mathrm{TT}ij}\epsilon_{ca(i}n_{j)}n^a = -2\dot{h}^{\mathrm{TT}ij}\epsilon_{cai}n_jn^a = 0 \qquad (20)$$

since $h^{\mathrm{TT}ij}n_j = 0$. We also find (using this orthogonality relation)

$$\begin{aligned}
\mathrm{Tr}[(\mathcal{L}_c\mathbf{\Pi})\ddot{\mathbf{Q}}\dot{\mathbf{h}}^{\mathrm{TT}}] &= -2\epsilon_{ca(i}n_{j)}n^a\ddot{Q}^j{}_k\dot{h}^{\mathrm{TT}ki} \\
&= -\epsilon_{cai}n_jn^a\ddot{Q}^j{}_k\dot{h}^{\mathrm{TT}ki} \\
&= -\frac{2}{R}\epsilon_{cai}n_jn^a\ddot{Q}^j{}_k\left[\Pi^k{}_l\ddot{Q}^l{}_m\Pi^{mi} - \frac{1}{2}\Pi^{ki}\mathrm{Tr}(\mathbf{\Pi}\ddot{\mathbf{Q}})\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{R}\epsilon_{cai}n_j n^a \ddot{Q}^j{}_k \ddot{Q}^l{}_m \left(\Pi^k{}_l \Pi^{mi} - \frac{1}{2}\Pi^{ki}\Pi^l{}_m\right) \\
&= -\frac{2}{R}\epsilon_{cai}n_j n^a \ddot{Q}^j{}_k \ddot{Q}^l{}_m \left[(\delta^k{}_l - n^k n_l)(\delta^{mi} - n^m n^i) - \frac{1}{2}(\delta^{ki} - n^k n^i)(\delta_l{}^m - n_l n^m)\right] \\
&= -\frac{2}{R}\epsilon_{cai}n_j n^a \ddot{Q}^j{}_k \ddot{Q}^l{}_i(\delta^k{}_l - n^k n_l) - \frac{1}{R}\epsilon_{cai}n_j n^a \ddot{Q}^{ji}\ddot{Q}^{lm} n_l n_m \\
&= -\frac{2}{R}\epsilon_{cai}n_j n^a \ddot{Q}^j{}_k \ddot{Q}^l{}_i(\delta^k{}_l - n^k n_l) - \frac{1}{R}\epsilon_{cai}n_j n^a \ddot{Q}^{ji}\ddot{Q}^{lm} n_l n_m. \qquad (21)
\end{aligned}$$

Comparing to Eq. (19), we see that the emitted angular momentum is

$$\mathcal{H}_c = \epsilon_{cai}\langle \ddot{Q}^j{}_k \ddot{Q}^l{}_i\rangle \left\langle n_j n^a(\delta^k{}_l - n^k n_l)\right\rangle_{\boldsymbol{n}} + \frac{1}{2}\epsilon_{cai}\langle \ddot{Q}^{ji}\ddot{Q}^{lm}\rangle \left\langle n_j n^a n_l n_m\right\rangle_{\boldsymbol{n}}, \qquad (22)$$

where the $\langle\rangle_{\boldsymbol{n}}$ averages are taken over directions. They simplify to

$$\left\langle n_j n^a(\delta^k{}_l - n^k n_l)\right\rangle_{\boldsymbol{n}} = \frac{4}{15}\delta^a_j\delta^k_l - \frac{1}{15}(\delta^a_l\delta^k_j + \delta^{ak}\delta_{jl}) \quad \text{and} \quad \left\langle n_j n^a n_l n_m\right\rangle_{\boldsymbol{n}} = \frac{1}{15}(\delta^a_j\delta^l_m + \delta^a{}_l\delta_{jm} + \delta^a_m\delta_{lj}). \qquad (23)$$

Here the δ^k_j term (1st average) and δ^l_m term (2nd average) will have no effect since $\mathbf{Q}$ is traceless. The remaining terms collect to give

$$\begin{aligned}
\mathcal{H}_c &= \frac{4}{15}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle - \frac{1}{15}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle + \frac{1}{30}\epsilon_{cai}\langle \ddot{Q}^{ji}\ddot{Q}^a{}_j\rangle + \frac{1}{30}\epsilon_{cai}\langle \ddot{Q}^{ji}\ddot{Q}_j{}^a\rangle \\
&= \frac{4}{15}\epsilon_{cai}\langle \dddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle - \frac{2}{15}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle \\
&= \frac{4}{15}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle - \frac{2}{15}\epsilon_{cai}\left\langle \frac{d}{dt}(\ddot{Q}^a{}_l \dot{Q}^l{}_i)\right\rangle + \frac{2}{15}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle \\
&= \frac{2}{5}\epsilon_{cai}\langle \ddot{Q}^a{}_l \ddot{Q}^l{}_i\rangle, \qquad (24)
\end{aligned}$$

where we have used the assumption of no long-term drift of $\ddot{Q}^a{}_l\dot{Q}^l{}_i$ to eliminate the total derivative term.

This is our final result for the total rate of emission of angular momentum:

$$\mathcal{H}_c = \frac{2}{5}\epsilon_{cai}\langle \ddot{Q}^a{}_l \dddot{Q}^{li}\rangle. \qquad (25)$$

III. ENERGY OF A BINARY SYSTEM

In order to proceed to the evolution of a binary system, we must determine its total energy. Put simply: given a collection of objects $\{A\}$ of mass m_A – for example, stars in a galaxy, or more fundamentally electrons and ions in a star – how does one determine the total mass? In Newtonian gravity, the answer is

$$M = \sum_A m_A, \qquad (26)$$

but we will see in general relativity that it is not.

In order to make life simpler, we will make the following assumptions:

- The self-gravity of individual consituents can be neglected. This is a good approximation for e.g. a proton in the Sun, whose potential well depth $m_p/r_p = Gm_p/r_pc^2 \sim 10^{-43}$, versus the Sun's potential well depth of $GM_\odot/R_\odot c^2 = 2\times 10^{-6}$. (Of course, if the constituents are really pointlike particles, this description doesn't work. In full GR, the only consistent description of a "point" particle is a black hole. In the case of an elementary particle of mass $< m_{\rm Planck} \sim 10^{-5}$ g, there is no meaningful description of its gravitational field at distances smaller than the Compton wavelength.)
- The objects are slowly moving and gravity is weak. In particular, we work to order v^2 in the velocities and Φ in the potential. This makes sense because for virialized systems, v^2 and Φ are typically of the same order.

The total mass of the system is simply its energy measured in the center of mass frame. That is, we are interested in the energy

$$E = \int T^{\mathrm{eff}\,00}\, d^3x \tag{27}$$

and momentum $P^i = \int T^{\mathrm{eff}\,0i}\, d^3x$. Then the mass is

$$M = \sqrt{E^2 - (P^i)^2}. \tag{28}$$

We note that in the "naive" center-of-mass frame, obtained by setting

$$\sum_A m_A \boldsymbol{v}_A = 0, \tag{29}$$

the momentum becomes at least of order $\sim Mv^2$, so $(P^i)^2$ is of order $\sim M^2v^4$. Therefore it is of high order and we may take

$$M = E = \int T^{\mathrm{eff}\,00}\, d^3x = \int T^{00}\, d^3x + \int t^{00}\, d^3x. \tag{30}$$

This integral manifestly has two parts: an integral of the 00 component of the stress-energy tensor over the coordinate volume, and an integral over the effective gravitational energy.

A. The first integral

To obtain the first integral in Eq. (30), we need the formula for the energy density of a particle. In special relativity, this was

$$T^{00}(t, x^i) = p^0 \delta^{(3)}[x^i - y^i(t)] = \frac{m_A}{\sqrt{1 - \boldsymbol{v}_A^2}} \delta^{(3)}[x^i - y_A^i(t)]. \tag{31}$$

In GR this gets modified. The above equation should be true in a local Lorentz frame – *any* local Lorentz frame – but the coordinate frame is not of this type. Instead, we recall the metric for a system of slow-moving particles to first order,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{32}$$

Then at the instantaneous position of a particle, we may define the local Lorentz frame of an observer at rest relative to the coordinate system,

$$\hat{t} = (1 + \Phi)(t - t_{\mathrm{orig}}), \quad \hat{x}^i = (1 - \Phi)(x^i - x^i_{\mathrm{orig}}), \tag{33}$$

where one can readily see that at the origin $(t_{\mathrm{orig}}, x^i_{\mathrm{orig}})$ the metric is $-d\hat{t}^2 + (d\hat{x}^i)^2$. If we choose the spatial origin of the local Lorentz frame to be at the position of the particle at some time t_{orig}, then we have

$$T^{\hat{0}\hat{0}} = \frac{m_A}{\sqrt{1 - \boldsymbol{v}_A^2}} \delta^{(3)}(\hat{x}^i). \tag{34}$$

Converting to the original coordinate system involves two factors of $dt/d\hat{t} = 1 - \Phi$, and the Jacobian $|d^3x^i/d^3\hat{x}^j| = 1 + 3\Phi$. Furthermore, to relevant order we may Taylor-expand the inverse square root, yielding

$$T^{00}(t, x^i) = m_A \left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2 + \Phi\right] \delta^{(3)}[x^i - y_A^i(t)]. \tag{35}$$

Integration is trivial due to the δ-function, and we get that the first integral is

$$\int T^{00}\, d^3x = \sum_A m_A \left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2 + \Phi(\boldsymbol{y}_A)\right]. \tag{36}$$

At first sight this looks like the Newtonian result. But it is not: writing the gravitational potential in terms of a sum over the sources, we find

$$\int T^{00}\, d^3x = \sum_A m_A + \sum_A \frac{1}{2} m_A (\boldsymbol{v}_A)^2 - \sum_{A,B} \frac{m_A m_B}{r_{AB}} : \tag{37}$$

the gravitational potential energy term is double-counted. (It is also infinite for true point masses, but if we imagine a continuous distribution of mass it is finite. This problem is no different in linearized GR than it is in Newtonian physics – the difference is that full nonlinear GR provides a way to make sense of a point mass as a black hole.) The solution to the double-counting problem is to return to the "gravitational energy" term.

B. The second integral

The effective gravitational energy can be obtained from the previous lecture, using $\bar{h}^{00} = -4\Phi$, $\bar{h} = 4\Phi$, and other components vanishing. From Lecture XI, however, and in the limit of nonrelativistic sources where $\Box \rightarrow \nabla^2$ (slowly varying or $\partial_t^2 \rightarrow 0$ approximation):

$$\int t^{00}\, d^3x^i = -\frac{1}{8\pi} \int \Phi \nabla^2 \Phi\, d^3x^i = -\frac{1}{2} \int \rho\Phi\, d^3x^i = -\frac{1}{2} \sum_A m_A \Phi(\boldsymbol{y}_A) = \frac{1}{2} \sum_{A,B} \frac{m_A m_B}{r_{AB}}. \tag{38}$$

Putting this together with Eq. (37) yields a total mass:

$$M = \sum_A m_A + \sum_A \frac{1}{2} m_A (\boldsymbol{v}_A)^2 - \frac{1}{2} \sum_{A,B} \frac{m_A m_B}{r_{AB}}. \tag{39}$$

This solves the double-counting problem from the first integral: Eq. (39) is (aside from the troublesome self-energy term in the sum where $A = B$, which does not exist for continuous mass distributions) the sum of the masses plus the Newtonian expression for the kinetic and potential energies.

We may thus conclude that the total mass of e.g. a binary star is the sum of the masses plus the standard Newtonian orbital energy, $-m_A m_B/(2a)$, where a is the semimajor axis. Thus **the negative "gravitational energy" of any system – a planet, star, or multiple system – itself gravitates. If the system is bound, its total mass is less than the sum of its parts**.

IV. APPLICATION: INSPIRAL OF A BINARY STAR

As a final application, let us consider the evolution of a binary star composed of two components with masses M_1 and M_2 with separation a on a circular orbit. We will make the velocities involved nonrelativisitic. The system has a kinetic+potential energy of

$$E_{\rm orb} = -\frac{M_1 M_2}{2a} \tag{40}$$

and hence a total mass of

$$M = M_1 + M_2 - \frac{M_1 M_2}{2a}. \tag{41}$$

The orbital frequency of the system is

$$\Omega \equiv \frac{2\pi}{P} = \frac{(M_1 + M_2)^{1/2}}{a^{3/2}}. \tag{42}$$

Our interest is in following the effect of gravitational radiation on the orbit. To do this, we first need to find the quadrupole moment. For masses separated at angle $\phi = \phi_0 + \Omega t$, this is

$$Q_{ij} = \frac{M_1 M_2}{M_1 + M_2} a^2 \begin{pmatrix} \cos^2\phi - \frac{1}{3} & \cos\phi \sin\phi & 0 \\ \cos\phi \sin\phi & \sin^2\phi - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{43}$$

If we use the double-angle identities, this becomes

$$Q_{ij} = \frac{M_1 M_2}{M_1 + M_2} a^2 \begin{pmatrix} \frac{1}{2}\cos 2\phi + \frac{1}{6} & \frac{1}{2}\sin 2\phi & 0 \\ \frac{1}{2}\sin 2\phi & -\frac{1}{2}\cos 2\phi - \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \tag{44}$$

and taking the third derivative gives

$$\dddot{Q}_{ij} = \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega^3 \begin{pmatrix} 4\sin 2\phi & 4\cos 2\phi & 0 \\ 4\cos 2\phi & -4\sin 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

The gravitational wave power is then

$$-\langle \dot{E} \rangle = \frac{1}{5}\langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle = \frac{1}{5}\left(\frac{M_1 M_2}{M_1 + M_2} a^2 \Omega^3 \right)^2 [32] = \frac{32}{5}\left(\frac{M_1 M_2}{M_1 + M_2} \right)^2 a^4 \Omega^6. \tag{46}$$

Using Kepler's second law to eliminate Ω gives

$$-\langle \dot{E} \rangle = \frac{32}{5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{a^5}. \tag{47}$$

This is the rate at which the system loses orbital energy. Assuming that the masses of the objects don't change (e.g. that there is no transfer of energy from the internal structure of the bodies into the orbit), we may equate this with the rate of change of orbital energy,

$$\langle \dot{E} \rangle = \partial_t \left(M_1 + M_2 - \frac{M_1 M_2}{2a} \right) = \frac{M_1 M_2}{2a^2} \dot{a}, \tag{48}$$

and hence obtain

$$\dot{a} = -\frac{64}{5} \frac{M_1 M_2 (M_1 + M_2)}{a^3}. \tag{49}$$

The $-$ sign indicates that the two bodies spiral together.

Since the rate of inspiral due to gravitational wave emission is proportional to a^{-3}, it follows that as the two bodies approach each other, they inspiral faster and faster. One may find the approach time by taking

$$\partial_t (a^4) = 4a^3 \dot{a} = -\frac{256}{5} M_1 M_2 (M_1 + M_2), \tag{50}$$

and hence we see that the inspiral reaches $a = 0$ in a finite time

$$t_{\rm GW} = \frac{5a^4}{256 M_1 M_2 (M_1 + M_2)}. \tag{51}$$

This time is shortest for massive bodies on close orbits, as one might expect.

A. Examples

As a simple example, let's consider the inspiral times associated with solar-system scales. Recall that, converted into times, a solar mass is 4.9 μs and the astronomical unit is 500 s. Therefore, we can calculate the inspiral time of a system:

$$t_{\rm GW} = 3.3 \times 10^{17}\,\text{yr}\, \frac{(a/1\,\text{AU})^4}{M_1 M_2 (M_1 + M_2)/M_\odot^3}. \tag{52}$$

For the Earth orbiting the Sun, with $M_1 = M_\odot$ and $M_2 = 3 \times 10^{-6} M_\odot$ at a separation of 1 AU, the inspiral time is 10^{23} years. Of course by then the Sun will have turned into a white dwarf, Mercury and (maybe) Venus and Earth will have been consumed, and it is doubtful even that the orbits of the other planets are stable over that timescale. As a more extreme example one could consider the "hot Jupiters" that have been found around other stars with

$M_1 \sim 10^{-3} M_\odot$ and $a = 0.05$ AU. There the inspiral time is 2×10^{15} years. So we can see that even in extreme situations, gravitational waves have no effect on planetary orbits.

Gravitational waves do however have a more significant effect on binary stars. If we consider a binary with masses of $M_1 = M_2 = M_\odot$, and we ask how close the orbits must be to merge in less than the age of the Universe (10^{10} years), we find

$$a < 0.016\,\mathrm{AU} \quad \text{or} \quad P < 12\,\mathrm{hr}. \tag{53}$$

There are many instances of stellar remnants (white dwarfs and neutron stars) in orbits with periods of this order of magnitude or shorter (even as short as a few minutes). Such objects will spiral in due to gravitational wave emission. In the case of binary neutron star mergers, these will be detectable as bursts of gravitational waves by the next generation of detectors. In the case of systems involving white dwarfs other outcomes are possible depending on the mass ratio: eventually the lower-mass star will fill its Roche lobe (the region where material can be bound to it rather than the other star), and either stable mass transfer or the tidal disruption of the lower-mass star will occur.