

Lecture XI: Asymptotically flat spacetimes and gravitational energy

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I. OVERVIEW

The previous discussions have taken place in the context of linearized GR, which is not a fully consistent theory. We will now discuss some aspects of GR in the nonlinear regime, with particular attention to isolated systems. We will learn that:

- For an isolated system, it is possible to define the total energy, momentum, mass dipole moment, and angular momentum based on surface integrals far from the system. This remains true even if the system contains strong gravitational fields (e.g. a neutron star).
- Due to nonlinearities in the GR field equations, curved spacetime is endowed with a “gravitational stress-energy” at second and higher order in perturbation theory.
- The usual conservation laws of energy, momentum, and angular momentum hold for such isolated systems, with corrections if gravitational waves are emitted associated with the flux of these quantities carried by gravitational waves.

The recommended reading for this lecture is:

- MTW §19.3, 20.1–20.3.

II. MASS AND ANGULAR MOMENTUM OF AN ISOLATED SYSTEM

In the previous discussion on linearized, we showed that the metric at large distances from an isolated system in linearized gravity could be written as

$$ds^2 = - \left(1 - 2\frac{M}{R}\right) dt^2 + 4\epsilon_{ijk} \frac{n_j S_k}{R^2} dx^i dt + \left(1 + 2\frac{M}{R}\right) [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + [\text{gravitational wave terms}], \quad (1)$$

where the gravitational wave terms decay as $\sim 1/R$. We want to establish how quantities like the “total energy” M can be determined in an asymptotically flat spacetime.

A. Pseudotensors and Gaussian integrals

Let us consider the object

$$H^{\mu\alpha\nu\beta} \equiv -\bar{h}^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\nu}\bar{h}^{\alpha\beta} + \bar{h}^{\alpha\nu}\eta^{\mu\beta} + \bar{h}^{\mu\beta}\eta^{\alpha\nu}. \quad (2)$$

As this is a perturbation on a Minkowski background, we will raise and lower indices on $H^{\mu\alpha\nu\beta}$ with respect to $\eta^{\mu\nu}$. This object satisfies the following symmetry properties:

$$\begin{aligned} H^{(\mu\alpha)\nu\beta} &= 0 && \text{antisymmetric on first two indices,} \\ H^{\mu\alpha(\nu\beta)} &= 0 && \text{antisymmetric on last two indices,} \\ H^{\mu\alpha\nu\beta} &= H^{\nu\beta\mu\alpha} && \text{symmetric under interchange of first and last two indices, and} \\ H^{\mu[\alpha\nu\beta]} &= 0 && \text{Jacobi identity.} \end{aligned} \quad (3)$$

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So far these statements do *not* depend on linear perturbation theory; they are just definitions. In linear theory, however, we have

$$H^{\mu\alpha\nu\beta}_{,\alpha\beta} = 2G^{\mu\nu} = 16\pi T^{\mu\nu}. \quad (4)$$

This enables us to write some integrals for the total momentum in a volume \mathcal{V} :

$$\begin{aligned} P^\mu &= \int_{\mathcal{V}} T^{\mu 0} d^3x \\ &= \frac{1}{16\pi} \int_{\mathcal{V}} H^{\mu\alpha 0\beta}_{,\alpha\beta} d^3x \\ &= \frac{1}{16\pi} \int_{\mathcal{V}} H^{\mu\alpha 0j}_{,\alpha j} d^3x \\ &= -\frac{1}{16\pi} \int_{\partial\mathcal{V}} H^{\mu\alpha 0j}_{,\alpha} n_j d^2x \end{aligned} \quad (5)$$

We may also determine the mass dipole moment:

$$\begin{aligned} \Delta^i &= \int_{\mathcal{V}} x^i T^{00} d^3x \\ &= \frac{1}{16\pi} \int_{\mathcal{V}} x^i H^{0j0k}_{,jk} d^3x \\ &= \frac{1}{16\pi} \left[\int_{\partial\mathcal{V}} x^i H^{0j0k}_{,k} n_j d^2x - \int_{\mathcal{V}} H^{0i0k}_{,k} d^3x \right] \\ &= \frac{1}{16\pi} \left[\int_{\partial\mathcal{V}} x^i H^{0j0k}_{,k} n_j d^2x + \int_{\partial\mathcal{V}} H^{0i0k} n_k d^3x \right], \end{aligned} \quad (6)$$

and a similar relation can be derived for the angular momentum:

$$S_i = \epsilon_{ijk} \int_{\mathcal{V}} x^j T^{0k} d^3x = \frac{1}{16\pi} \epsilon_{ijk} \left[\int_{\partial\mathcal{V}} x^j H^{0lk\beta}_{,\beta} n_l d^2x + \int_{\partial\mathcal{V}} H^{0jkl} n_l d^2x \right]. \quad (7)$$

We thus see that the low-order moments of the system that are either conserved or have simple evolution laws ($\dot{\Delta}^i = P^i$) are describable in terms of surface integrals.

B. Application to asymptotically flat spacetimes

In the limit of an asymptotically flat spacetime, it should be permissible to use the surface integrals, Eqs. (5,6,7) far from the source. It is therefore possible to speak of the *mass*, *momentum*, *center of mass*, and *angular momentum* of such a source, even if it contains strong gravitational fields, or even if it is a black hole.

In such situations, it is useful to define the effective stress-energy pseudotensor $T^{\text{eff}\mu\nu}$ by

$$T^{\text{eff}\mu\nu} \equiv \frac{1}{16\pi} H^{\mu\alpha\nu\beta}_{,\alpha\beta}. \quad (8)$$

In linearized GR, this is exactly equal to the stress-energy tensor. In full GR, it is not: the difference is called the *gravitational stress-energy pseudotensor* $t^{\mu\nu}$:

$$t^{\mu\nu} \equiv T^{\text{eff}\mu\nu} - T^{\mu\nu}. \quad (9)$$

Note that $t^{\mu\nu}$ is *not* a tensor: under general coordinate transformations, it has no reason to be well-behaved. However, it does transform in the usual way under **global** Lorentz transformations of the background.

Warning: Once again, $T^{\text{eff}\mu\nu}$ and $t^{\mu\nu}$ are not tensors. It is **not** possible to take the energy and angular momentum of an object with strong gravity and “localize” the part associated with the gravitational field; such a description cannot be gauge-invariant. However, the overall integrals are gauge-invariant.

The antisymmetry relations imply that

$$T^{\text{eff}\mu\nu}_{,\nu} = \frac{1}{16\pi} H^{\mu\alpha\nu\beta}_{,\alpha\beta\nu} = 0. \quad (10)$$

This is a sort of conservation law for the effective stress-energy pseudotensor. It is distinct from the law obeyed by the true stress-energy tensor $T^{\mu\nu}_{;\nu} = 0$.

The point of the stress-energy pseudotensor is that even for sources containing strong gravitational fields, it integrates to the proper momentum, center of mass, and angular momentum. For example, the relation

$$P^\mu = \int_{\mathcal{V}} T^{\text{eff} \mu 0} d^3x \quad (11)$$

where P^μ is defined for a self-gravitating object by Eq. (5) remains valid even for a neutron star. The conservation law, Eq. (10) then implies the usual relations:

- For an isolated system with no emerging gravitational radiation, P^μ is conserved.
- For an isolated system with no emerging gravitational radiation, the rate of change of the mass dipole moment $\dot{\Delta}^i = P^i$.
- For an isolated system with no emerging gravitational radiation, S_i is conserved.

By “no emerging gravitational radiation” we mean to set $t^{\mu\nu}$ to be negligible at large distances (see below).

We haven’t proven these rules for a black hole or other system with “weird” topology (such that you can’t actually do an integral of T^{eff} through the center of the object). Nevertheless the integral definitions combined with the equivalent conservation rule $H^{\mu\alpha\nu\beta}_{;\alpha\beta\nu}$ are sufficient to prove this. (Homework exercise!)

For a system emitting gravitational radiation, the situation is more complex. The Einstein tensor is now given not by $\frac{1}{2}H^{\mu\alpha\nu\beta}_{;\alpha\beta}$, but by higher-order terms as well:

$$G^{\mu\nu} = \frac{1}{2}H^{\mu\alpha\nu\beta}_{;\alpha\beta} - 8\pi t^{\mu\nu}, \quad (12)$$

where $t^{\mu\nu}$ contains all terms 2nd order and higher in $\bar{h}^{\mu\nu}$. The amplitude of emitted gravitational waves is $\propto 1/R$ and hence the gravitational stress-energy pseudotensor, being a second-order object, is $\propto 1/R^2$. This is important because if we write the time derivative of e.g. the system’s energy, and assume no matter is emitted ($T^{\mu\nu} = 0$ on the boundary of the region considered)

$$\dot{E} = \int_{\mathcal{V}} \dot{T}^{\text{eff} 00} d^3x = - \int_{\mathcal{V}} T^{\text{eff} 0i} n_i d^3x = - \int_{\partial\mathcal{V}} T^{\text{eff} 0i} n_i d^2x = - \int_{\partial\mathcal{V}} t^{0i} n_i d^2x. \quad (13)$$

For the case of emerging gravitational radiation, the latter integral approaches a constant as we take the surrounding surface to ∞ . Therefore a system can change its total energy through the emission of such waves. Similar rules tell us that it can emit angular momentum.

C. Further comments on the conservation of energy and momentum

We are now ready to understand how the local conservation law $T^{\mu\nu}_{;\nu} = 0$ relates to global conservation of energy and momentum and the concept of “gravitational energy.”

Taking the divergence of Eq. (9) and using $T^{\text{eff} \mu\nu}_{;\nu} = 0$ and $T^{\mu\nu}_{;\nu} = 0$ gives

$$t^{\mu\nu}_{;\nu} = T^{\mu\nu}_{;\nu} - T^{\mu\nu}_{;\nu} = \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\nu} T^{\mu\alpha}. \quad (14)$$

Considering first the $\mu = 0$ part of this, the left-hand side can be interpreted as the “rate of generation of gravitational energy per unit (coordinate) 4-volume.” It is not gauge-invariant. The right-hand side is the product of Christoffel symbols and stress-energy tensors: it represents the fact that in curved spacetime – *or even a curved coordinate system!* – different components of the matter momentum p^μ can be rotated into each other as the coordinate system changes. The Christoffel symbol simply represents the rate at which the coordinate system changes the definition of “energy” and causes the globally measured energy of the matter fields to decrease.

In order to understand the quantitative implications of all this, we need to develop the formula for $t^{\mu\nu}$, and determine the “effective” energy carried by gravitational waves. The computation of $t^{\mu\nu}$ to second order will be our next order of business.

III. FORMULA FOR THE GRAVITATIONAL STRESS-ENERGY PSEUDOTENSOR TO SECOND ORDER

Now that we have all of this nice theory, it is time to actually find the formula for $t^{\mu\nu}$. We will do this only to second order in the metric, since the first-order part vanishes. Second order will however be sufficient for computations of leading-order relativistic effects. We will further work only in the Lorentz gauge in order to simplify the possible expressions.

We will also compute only the average of this pseudotensor over some region of spacetime – specifically, we will drop terms of the form $s^{\mu\nu\gamma}{}_{,\gamma}$ where $s^{\mu\nu}$ is composed only of the metric perturbation and its derivatives. The logic is that while this does not give the correct local answer for $t^{\mu\nu}$, it gives the correct answer for bulk integrals (e.g. the total energy in a gravitational wave packet; the gravitational energy of an isolated system such as the Solar System or a neutron star, etc.). Since we work only to second order, and $t^{\mu\nu}$ is already second order, we do not need to worry about the exact nature of the averaging, which is ambiguous at third order.

A. General considerations

Let us begin by considering the possible forms of the answer. The Christoffel symbols $\Gamma^\nu{}_{\alpha\beta}$ contain several terms, each containing exactly one derivative of the metric. Then the Riemann tensor, and all contractions thereof, contain terms of the form $\partial\Gamma$ and $\Gamma\Gamma$ (derivatives and products of the Christoffel symbols), so the Einstein tensor $G^{\mu\nu}$ ultimately contains combinations of metric perturbations $h_{\mu\nu}$ that have exactly 2 derivatives. Since $t^{\mu\nu}$ vanishes to first order by construction, at second order we have only perturbations with 2 factors of h (or \bar{h}) and 2 derivatives. This implies

$$t^{\mu\nu} = t^{\mu\nu}_{(\mathbf{C})} + t^{\mu\nu}_{(\mathbf{D})} = C^{\mu\nu\alpha\beta\gamma\delta\epsilon\zeta} \bar{h}_{\alpha\beta} \bar{h}_{\gamma\delta,\epsilon\zeta} + D^{\mu\nu\alpha\beta\gamma\delta\epsilon\zeta} \bar{h}_{\alpha\beta,\epsilon} \bar{h}_{\gamma\delta,\zeta}, \quad (15)$$

where \mathbf{C} and \mathbf{D} are 8th rank tensors that don't depend on $\bar{h}_{\mu\nu}$. This sounds awful (they each have $4^8 = 65536$ components, for a total of 131072) but in fact they must be constructed out of the metric $\eta^{\mu\nu}$. There are $8!/(2!^4 \cdot 4!) = 105$ ways to do this (they are permutations of the indices on $\eta^{\mu\nu}\eta^{\alpha\beta}\eta^{\gamma\delta}\eta^{\epsilon\zeta}$ – and there are 105 ways to break 8 indices into 4 pairs). Thus the entire problem of the gravitational stress-energy pseudotensor is reduced to the 210 coefficients in \mathbf{C} and \mathbf{D} .

Now the neglect of surface terms – i.e. dropping terms that are schematically of the form $\partial(h\partial h)$ – allows us to integrate by parts and convert all the \mathbf{D} -type terms into \mathbf{C} -type terms. So we don't need \mathbf{D} , and $t^{\mu\nu}$ now depends only on 105 coefficients.

In fact, further simplification is possible. Consider \mathbf{C} first. For the possible terms where we expand \mathbf{C} as 4 factors of η , there are several restrictions and symmetries:

- We cannot pair ϵ or ζ with γ or δ .
- There must be symmetry under interchange of μ and ν .
- Terms obtained by switching $\alpha \leftrightarrow \beta$, $\gamma \leftrightarrow \delta$, and $\epsilon \leftrightarrow \zeta$ are redundant due to the symmetries of $\bar{h}_{\alpha\beta}\bar{h}_{\gamma\delta,\epsilon\zeta}$.
- Double integration by parts enables us to switch $\alpha\beta$ with $\gamma\delta$: $h_{\alpha\beta}h_{\gamma\delta,\epsilon\zeta} \sim -h_{\alpha\beta,\zeta}h_{\gamma\delta,\epsilon} \sim h_{\alpha\beta,\zeta\epsilon}h_{\gamma\delta}$. A corollary is that we cannot pair ϵ or ζ with α or β .

This leaves us with only 6 independent coefficients in \mathbf{C} :

$$\langle t^{\mu\nu} \rangle = \left\langle c_1 \bar{h} \bar{h}{}^{\mu\nu} + c_2 \bar{h}_{\alpha\gamma} \bar{h}^{\alpha\gamma,\mu\nu} + c_3 \eta^{\mu\nu} \bar{h} \square \bar{h} + c_4 \eta^{\mu\nu} \bar{h}_{\alpha\beta} \square \bar{h}^{\alpha\beta} + c_5 \bar{h}^{\mu\nu} \square \bar{h} + c_6 \bar{h}^{\beta(\mu} \square \bar{h}^{\nu)}{}_{\beta} \right\rangle. \quad (16)$$

B. Finding the coefficients

Of course, to do any actual calculation we need $c_1 \dots c_6$. At first it sounds like we are back in our original situation – we need a messy calculation. But this is not so. Some simple arguments and two example cases enable us to find all of the coefficients.

First, let us recall that to first order we may write a “pure gauge” gravitational wave $\xi^\alpha = \Re(iB^\alpha e^{ik_\sigma x^\sigma})$ with \mathbf{k} null, which (by virtue of satisfying $\square \xi^\alpha = 0$) preserves the Lorentz gauge. We then have

$$\bar{h}_{\mu\nu} = (k_\mu B_\nu + k_\nu B_\mu - k_\alpha B^\alpha \eta_{\mu\nu}) \cos(k_\sigma x^\sigma). \quad (17)$$

This is to first order, but since we are displacing the coordinate system it must be that this $\bar{h}_{\mu\nu}$ is the first-order term in some more general description of flat spacetime. In this case we must have $t^{\mu\nu} = 0$, and so – plugging this metric perturbation into Eq. (16) and noting that $\square \bar{h}_{\mu\nu} = 0$ and that $\bar{h} = -2k_\alpha B^\alpha \cos(k_\sigma x^\sigma)$ – we find

$$\langle [-c_1 k^\mu k^\nu 4(k_\alpha B^\alpha)^2 - c_2 k^\mu k^\nu 2(k_\alpha B^\alpha)^2] \cos^2(k_\sigma x^\sigma) \rangle = 0, \quad (18)$$

which requires $c_2 = -2c_1$.

To go further, we must resort to some specific examples of perturbations. To keep things simple, let us consider the following perturbation:

$$\bar{h}_{12} = 2b \cos x^0, \quad \text{all other terms vanish}, \quad (19)$$

and consider the Einstein tensor to order b^2 . The metric coefficients are $\eta_{\mu\nu}$ except for $g_{12} = 2b \cos x^0$. The nonzero Christoffel symbols are

$$\Gamma^0_{(12)} = -b \sin x^0, \quad \Gamma^1_{(02)} = \Gamma^2_{(01)} = -b \sin x^0, \quad \Gamma^1_{(01)} = \Gamma^2_{(02)} = 2b^2 \sin x^0 \cos x^0 \quad (20)$$

(the last exists because of $g^{12} = -2b \cos x^0$). The Einstein tensor through order b^2 can be obtained by the usual construction of the Ricci tensor, reversal of the trace to get the Einstein tensor, and raising of indices. The result is

$$G^{00} = b^2 \sin^2 x^0, \quad G^{11} = G^{22} = \frac{1}{2} b^2 \sin^2 x^0, \quad G^{12} = G^{21} = -b \cos x^0, \quad \text{and} \quad G^{33} = b^2(-4 + 7 \sin^2 x^0), \quad (21)$$

with other components zero. Doing a spacetime average gives

$$\langle G^{00} \rangle = -\frac{1}{2} b^2, \quad \langle G^{11} \rangle = \langle G^{22} \rangle = \frac{1}{2} b^2, \quad \langle G^{12} \rangle = \langle G^{21} \rangle = 0, \quad \langle G^{33} \rangle = -\frac{1}{2} b^2, \quad (22)$$

with the rest equal to zero. Now the spacetime averages in this example all vanish to linear order, so Eq. (16) must give the second-order result. We see that $\bar{h} = 0$ in this example and due to the sinusoidal nature $\square \rightarrow +1$, so we have

$$\begin{aligned} \langle t^{00} \rangle &= -\frac{1}{8\pi} \langle G^{00} \rangle = \frac{1}{16\pi} b^2 = -4b^2 c_2 - 4b^2 c_4; \\ \langle t^{11} \rangle &= -\frac{1}{8\pi} \langle G^{11} \rangle = -\frac{1}{16\pi} b^2 = 4b^2 c_4 + 2b^2 c_6; \\ \langle t^{12} \rangle &= -\frac{1}{8\pi} \langle G^{12} \rangle = 0 = 0; \quad \text{and} \\ \langle t^{33} \rangle &= -\frac{1}{8\pi} \langle G^{33} \rangle = \frac{1}{16\pi} b^2 = 4b^2 c_4. \end{aligned} \quad (23)$$

This establishes from $\langle t^{33} \rangle$ the values of $c_4 = \frac{1}{64\pi}$; then from $\langle t^{00} \rangle$ the value $c_2 = -\frac{1}{32\pi}$ and hence (from the pure gauge mode before) $c_1 = \frac{1}{64\pi}$; and finally from $\langle t^{11} \rangle$ we find $c_6 = -\frac{1}{16\pi}$. Thus 4 of the coefficients are determined.

To establish the remaining two coefficients c_3 and c_5 , we must consider a non-traceless metric perturbation ($\bar{h} \neq 0$). The obvious candidate is

$$\bar{h}_{11} = \bar{h}_{22} = \bar{h}_{33} = 2f \cos x^0, \quad \text{all others vanish}, \quad (24)$$

with f small. This manifestly satisfies the Lorentz gauge. Then $\bar{h} = 6f \cos x^0$ and the normal metric perturbations are

$$h_{00} = 3f \cos x^0, \quad h_{ij} = -f \delta_{ij} \cos x^0, \quad \text{all others vanish}. \quad (25)$$

The nonzero Christoffel symbols are

$$\Gamma^0_{00} = \frac{3}{2} f \sin x^0 (1 + 3f \cos x^0), \quad \Gamma^0_{ij} = \frac{1}{2} f \delta_{ij} \sin x^0 (1 + 3f \cos x^0), \quad \Gamma^i_{(0j)} = \frac{1}{2} f \delta^i_j \sin x^0 (1 + f \cos x^0). \quad (26)$$

This time, computation of the Einstein tensor gives

$$G^{00} = \frac{3}{4} f^2 \sin^2 x^0 \quad \text{and} \quad G^{11} = G^{22} = G^{33} = -f \cos x^0 - 5f^2 \cos^2 x^0 + \frac{7}{4} f^2 \sin^2 x^0. \quad (27)$$

Averaging gives

$$\langle G^{00} \rangle = \frac{3}{8}f^2, \quad \langle G^{11} \rangle = -\frac{13}{8}f^2, \quad (28)$$

or $\langle t^{00} \rangle = -\frac{3}{64\pi}f^2$, $\langle t^{11} \rangle = \frac{13}{64\pi}f^2$. This is to be compared to Eq. (16), which gives

$$\langle t^{00} \rangle = -f^2(18c_1 + 6c_2 + 18c_3 + 6c_4) \quad \text{and} \quad \langle t^{11} \rangle = f^2(18c_3 + 6c_4 + 6c_5 + 2c_6). \quad (29)$$

We thus see that $c_3 = -\frac{1}{128\pi}$ and $c_5 = \frac{1}{16\pi}$.

The overall formula for the effective gravitational stress-energy tensor is then

$$\langle t^{\mu\nu} \rangle = \left\langle \frac{1}{64\pi} \bar{h} \bar{h}^{\mu\nu} - \frac{1}{32\pi} \bar{h}_{\alpha\gamma} \bar{h}^{\alpha\gamma, \mu\nu} - \frac{1}{128\pi} \eta^{\mu\nu} \bar{h} \square \bar{h} + \frac{1}{64\pi} \eta^{\mu\nu} \bar{h}_{\alpha\beta} \square \bar{h}^{\alpha\beta} + \frac{1}{16\pi} \bar{h}^{\mu\nu} \square \bar{h} - \frac{1}{16\pi} \bar{h}^{\beta(\mu} \square \bar{h}^{\nu)\beta} \right\rangle. \quad (30)$$

Remember though that this is only appropriate for use in averages or spatial integrals, and only to second order in perturbation theory.

IV. SPECIAL CASE: THE ENERGY OF GRAVITATIONAL WAVES

In this case, we consider a transverse-traceless metric perturbation \bar{h}_{ij}^{TT} . Then $\square \bar{h}_{ij} = 0$ and $\bar{h} = 0$ and we find

$$\langle t^{\mu\nu} \rangle = -\frac{1}{32\pi} \langle \bar{h}_{ij}^{\text{TT}} \bar{h}^{\text{TT} ij, \mu\nu} \rangle = \frac{1}{32\pi} \langle \bar{h}_{ij, \mu}^{\text{TT}} \bar{h}^{\text{TT} ij, \nu} \rangle. \quad (31)$$

We thus see that there is an effective energy density given by

$$\rho_{\text{GW}} \equiv \langle t^{00} \rangle = \frac{1}{32\pi} \langle \dot{\bar{h}}_{ij}^{\text{TT}} \dot{\bar{h}}^{\text{TT} ij} \rangle; \quad (32)$$

this is proportional to the square of the strain rate and independent of polarization. It is also positive-definite: gravitational waves carry positive energy. Furthermore, if the wave train is propagating in the 3-direction (i.e. a plane wave independent of x^1 and x^2 , and depending only on the combination $t - x^3$) then $\partial_3 = -\partial_0$ and we find

$$\langle t^{03} \rangle = \langle t^{33} \rangle = \rho_{\text{GW}}, \quad \langle t^{1\mu} \rangle = 0. \quad (33)$$

Thus there is no transverse momentum or stress, but there is an energy flux, 3-momentum density, and 3-momentum flux in the forward (x^3) direction. We thus learn that, like electromagnetic waves, gravitational waves carry momentum.

But this momentum is fundamentally different from others we have encountered: it arises due to the nonlinearity of Einstein's equations. Inside a gravitational wave train, over distances small compared to a wavelength, one may always find a local Lorentz frame, and one may measure “ ρ ” by looking at the relative accelerations of nearby test particles: the answer will be zero. The gravitational wave has $T^{\mu\nu} = 0$ everywhere. But viewed at the macroscopic scale, averaged over regions of many wavelengths, space that is empty aside from gravitational waves will become curved by the apparent presence of the source $t^{\mu\nu}$.