Lecture X: External fields and generation of gravitational waves

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I. OVERVIEW

Having examined weak field gravity and the associated experimental tests, we now turn our attention to the external field produced by a system (e.g. a star or the solar system) in linearized gravity. This will include both the gravitational analogues of electric and magnetic multipole moments, as well as gravitational waves.

The discussion in the beginning part of this lecture makes sense only for weak perturbations around Minkowski spacetime. Later we will generalize the concepts to make sense in *asymptotically flat* spacetime – i.e. spacetime that looks like Minkowski far from the system, but may have strongly curved regions inside of it (e.g. a black hole).

The recommended reading for this lecture is:

- MTW §19.1–19.2.

II. GREEN'S FUNCTIONS IN LORENTZ GAUGE

We found that in Lorentz gauge the trace-reversed metric perturbation is given via the relation

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{1}$$

We would like to formally solve this equation using a Green's function approach. That is, we wish to construct the *Green's function* $G(x^\alpha)$ such that

$$\Box G(x^\alpha) = \delta^{(4)}(x^\alpha), \tag{2}$$

and then by the principle of superposition we may write the metric perturbation as an integral over the Green's function:

$$\bar{h}_{\mu\nu}(x^\alpha) = -16\pi \int G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha)\, d^4\boldsymbol{y}. \tag{3}$$

How are we to find G? There is unfortunately no unique answer! After all, we could add any function f with $\Box f = 0$ to G and it would still satisfy Eq. (2). However, in most situations there is a physical choice: we want the *retarded Green's function* $G_{\rm ret}$, which is zero for $x^0 = t < 0$. This corresponds to the solution in which there is no incoming gravitational radiation. The use of the retarded Green's function is however not a necessity but a particular solution to Einstein's equations.

[Note: The notion of a retarded Green's function is slightly tricky for theories with a conserved source, since in any physical system there was nonzero $T^{\mu\nu}$ somewhere back into the infinite past. However it is clear that if we start with a stationary mass distribution and no incident gravitational waves, and then allow some complex dynamics to occur, then a retarded Green's function will respect causality and not radiate gravitational waves into the past.]

The easiest method to obtain the retarded Green's function is to expand Eq. (2) in spherical coordinates (t, r, θ, ϕ). We note that for a function G with spherical symmetry,

$$\Box G = -\partial_t^2 G + \nabla^2 G = -\partial_t^2 G + \frac{1}{r}\partial_r^2(rG) = \frac{1}{r}(-\partial_t^2 + \partial_r^2)(rG). \tag{4}$$

Since the 4-dimensional δ function can be written as

$$\delta^{(4)}(x^\alpha) = \delta(t)\delta(r-\epsilon)\frac{1}{4\pi\epsilon^2}, \tag{5}$$

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with the last factor introduced to give proper normalization for small but positive ϵ, we have

$$(-\partial_t^2 + \partial_r^2)(rG) = r\Box G = \delta(t)\delta(r-\epsilon)\frac{1}{4\pi\epsilon}. \tag{6}$$

This applies at $r > 0$; at $r = 0$ we have the boundary condition $rG = 0$. We may extend the range of validity to $r < 0$ by enforcing the boundary condition with a mirror source and ensuring the odd symmetry of rG:

$$(-\partial_t^2 + \partial_r^2)(rG) = \delta(t)\frac{\delta(r-\epsilon)-\delta(r+\epsilon)}{4\pi\epsilon} = -\frac{1}{2\pi}\delta(t)\delta'(r) = -\frac{1}{2\pi}\partial_r[\delta(t)\delta(r)]. \tag{7}$$

This equation can be solved via the definition $\Psi(t,r) = \int_{-\infty}^{r} rG\,dr$, so that $rG = \partial_r\Psi$. Then:

$$(-\partial_t^2 + \partial_r^2)\Psi = -\frac{1}{2\pi}\delta(t)\delta(r), \tag{8}$$

or switching to a characteristic coordinate system, $u = r - t$ and $v = t + r$,

$$4\frac{\partial^2\Psi}{\partial u\partial v} = -\frac{1}{\pi}\delta(u)\delta(v). \tag{9}$$

This has the solution

$$\Psi = \frac{1}{4\pi}\Theta(-u)\Theta(v) = \frac{1}{4\pi}\Theta(t-r)\Theta(t+r), \tag{10}$$

which is obviously retarded. Taking the derivative to get back rG yields the retarded Green's function

$$G_{\rm ret}(x^\alpha) = -\frac{1}{4\pi r}\delta(t-r)\Theta(t). \tag{11}$$

The retarded solution for the metric is then

$$\Box\bar{h}^{\rm ret}_{\mu\nu}(\boldsymbol{x}_{\rm sp}, t) = 4\int\frac{1}{r}T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-r)d^3\boldsymbol{y}_{\rm sp}, \tag{12}$$

where $r \equiv |\boldsymbol{x}_{\rm sp} - \boldsymbol{y}_{\rm sp}|$.

Remember that the retarded solution is only one possible solution for the metric for a given matter source. But the difference between the true solution and the retarded solution can be found by noting that if both are soutions of $\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$ for the same source, then

$$\Box(\bar{h}_{\mu\nu} - \bar{h}^{\rm ret}_{\mu\nu}) = 0. \tag{13}$$

Therefore, we may write

$$\bar{h}_{\mu\nu} = \bar{h}^{\rm ret}_{\mu\nu} + \bar{h}^{\rm homo}_{\mu\nu}, \tag{14}$$

where $\bar{h}^{\rm homo}_{\mu\nu}$ is a homogeneous or gravitational wave solution (solution for zero matter source). We will see that Eq. (12) contains gravitational waves, but they are only outgoing (as one can see from the $t - r$ time argument). Therefore setting $\bar{h}^{\rm homo}_{\mu\nu}$ to zero is equivalent to saying that no external gravitational waves are incident on a system. Usually this is a good approximation!

[**Warning**: The outgoing gravitational waves in the above formulation are not necessarily in transverse-traceless gauge.]

III. MULTIPOLE EXPANSION

Far from a source, it is common to do a *multipole expansion*: essentially a power-series expansion of Eq. (12) in powers of $\boldsymbol{y}_{\rm sp}$. This can be done for both relativistic and nonrelativistic sources, but we will focus on nonrelativistic sources (e.g. binary stars) here. We will also examine only the lowest-order multipoles since these (i) correspond to conserved quantities and (ii) the next-lowest terms carry the dominant source of gravitational radiation from a nonrelativistic object.

We will use (t, x^i) for the position of the point at which the metric is measured, y^i as the position of a point in the source, and then define $R = |\boldsymbol{x}_{\rm sp}|$, $n_i = x^i/R$, and $r = |\boldsymbol{x}_{\rm sp} - \boldsymbol{y}_{\rm sp}|$. The source is enclosed in a region of characteristic size $\sim L$ and evolves on a dynamical timescale $\sim t_{\rm dyn}$.

A. The trace-reversed perturbation

We begin by expanding $1/r$ as a power series in y:

$$\begin{aligned}\frac{1}{r} &= [(x_i - y_i)(x_i - y_i)]^{-1/2} \\ &= [x_i x_i - 2x_i y_i + y_i y_i]^{-1/2} \\ &= (x_i x_i)^{-1/2} - \frac{1}{2}(x_i x_i)^{-3/2}(-2x_i y_i + y_i y_i) + \frac{3}{8}(x_i x_i)^{-5/2}(2x_i y_i)^2 + ... \\ &= \frac{1}{R} - \frac{n_i y_i}{R^2} + \frac{n_i n_j (3y_i y_j - y_k y_k \delta_{ij})}{2R^3} + ..., \end{aligned} \tag{15}$$

with a fractional error of order $(L/R)^3$, where L is the typical size scale of the source.

We may also Taylor-expand $T_{\mu\nu}$:

$$T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-r) = T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-R) + n_i y_i T_{\mu\nu,0}(\boldsymbol{y}_{\rm sp}, t-R) + \frac{1}{2} n_i n_j y_i y_j T_{\mu\nu,00}(\boldsymbol{y}_{\rm sp}, t-R) ..., \tag{16}$$

with a fractional error of order $L^3/t_{\rm dyn}^3 = V^3$ or $L^2/Rt_{\rm dyn} = VL/R$, where V is the typical velocity scale of the source.

It then follows that the retarded solution for the metric perturbation is

$$\begin{aligned}\bar{h}_{\mu\nu}(\boldsymbol{x}_{\rm sp}, t) &= 4\int \left[\frac{1}{R} - \frac{n_i y_i}{R^2} + \frac{n_i n_j (3y_i y_j - y_k y_k \delta_{ij})}{2R^3} ...\right] \\ &\quad \times \left[T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-R) + n_i y_i \dot{T}_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-R) + \frac{1}{2} n_i n_j y_i y_j T_{\mu\nu,00}(\boldsymbol{y}_{\rm sp}, t-R)\right] d^3\boldsymbol{y}_{\rm sp}. \end{aligned} \tag{17}$$

The lowest-order fractional errors are V^2, VL/R, and $1/R^3$.

It is useful to consider the behavior of each metric component at large distances from the source. We first see that

$$\begin{aligned}\frac{1}{4}\bar{h}_{00}(\boldsymbol{x}_{\rm sp}, t) &= \frac{1}{R}\int \rho\, d^3\boldsymbol{y}_{\rm sp} - \frac{n_i}{R^2}\int y_i \rho\, d^3\boldsymbol{y}_{\rm sp} + \frac{n_i}{R}\int y_i \dot{\rho}\, d^3\boldsymbol{y}_{\rm sp} + \frac{3n_i n_j}{2R^3}\int (y_i y_j - \frac{1}{3} y_k y_k \delta_{ij})\rho\, d^3\boldsymbol{y}_{\rm sp} \\ &\quad + \frac{1}{2} n_i n_j \int y_i y_j \ddot{\rho}\, d^3\boldsymbol{y}_{\rm sp} + ... \Big|_{\rm ret}, \end{aligned} \tag{18}$$

where the $|_{\rm ret}$ on the right-hand side indicates evaluation at the retarded time $t - R$.

Now the first integral is simply the (conserved) mass M of the system. (Technically this is the energy, but the difference only arises at the next order in velocity, which we have dropped.) The second integral is the mass dipole moment,

$$MY_i = \int y_i \rho\, d^3\boldsymbol{y}_{\rm sp}, \tag{19}$$

where Y_i is the center of mass. Finally, defining the momentum density $F_j = T^0{}_j$ and noting that $F_{j,j} = -\dot{\rho}$ the third integral is

$$M\dot{Y}_i = \int y_i \dot{\rho}\, d^3\boldsymbol{y}_{\rm sp} = -\int y_i F_{j,j}\, d^3\boldsymbol{y}_{\rm sp} = \int y_{i,j} F_j\, d^3\boldsymbol{y}_{\rm sp} = \int F_i\, d^3\boldsymbol{y}_{\rm sp} = P_i, \tag{20}$$

where P_i is the total momentum. Under normal circumstances, we will work in the *center of mass frame*, in which Y_i and P_i are zero. Finally, we define the mass quadrupole moment of the system to be

$$Q_{ij} = \int (y_i y_j - \frac{1}{3} y_k y_k \delta_{ij})\rho\, d^3\boldsymbol{y}_{\rm sp} = I_{ij} - \frac{1}{3} I_{kk}\delta_{ij}. \tag{21}$$

(Here I_{ij} is the moment of inertia tensor.) Then Eq. (17) reduces to

$$\bar{h}_{00}(\boldsymbol{x}) = 4\frac{M}{R} + 6\frac{Q_{ij} n_i n_j}{R^3} + 2\frac{n_i n_j}{R}\ddot{I}_{ij} ... \Big|_{\rm ret}. \tag{22}$$

This is simply the standard quadrupolar formula familiar from Newtonian physics, plus a new term that contains $\ddot{I}$. **But** note that in fully nonlinear general relativity, the leading correction to the M/R formula is not the quadrupole term ($\propto R^{-3}$) but rather the nonlinearity of the theory ($\propto M^2/R^2$ – this is the term that is necessary to obtain the perihelion precession of Mercury). In some cases, such as the orbit of Mercury around the Sun, the relativistic M^2/R^2 term dominates over the quadrupole effect, but for a satellite orbiting the Earth (with its much more flattened shape) the opposite is true.

We may further write down the linearized formula for $\bar{h}_{0i}$, including terms through second order in V:

$$\bar{h}_{0i}(\boldsymbol{x}) = \frac{4}{R}\int F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} - 4\frac{n_j}{R^2}\int y_j F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} + 4\frac{n_j}{R}\int y_j \dot{F}_i\,d^3\boldsymbol{y}_{\rm sp} + ...\Big|_{\rm ret}; \tag{23}$$

the first integral is P_i and hence vanishes. The second integral may be simplified using the rule that the angular momentum is

$$S_k = -\epsilon_{ijk}\int y_j F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} \tag{24}$$

so that

$$\epsilon_{klm}S_k = -\epsilon_{klm}\epsilon_{ijk}\int y_j F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} = -(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\int y_j F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} = -2\int y_{[m}F_{l]}(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp}. \tag{25}$$

Furthermore,

$$\dot{I}_{ij} = \int y_i y_j \dot{\rho}\,d^3\boldsymbol{y}_{\rm sp} = -\int y_i y_j F_{k,k}\,d^3\boldsymbol{y}_{\rm sp} = \int (y_{i,k}y_jF_k + y_{j,k}y_iF_k)\,d^3\boldsymbol{y}_{\rm sp} = 2\int y_{(j}F_{i)}\,d^3\boldsymbol{y}_{\rm sp}. \tag{26}$$

Therefore, decomposing the integral in $\bar{h}_{0i}$ into its symmetric and antisymmetric parts:

$$-4\frac{n_j}{R^2}\int y_j F_i(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} = -4\frac{n_j}{R^2}\left[\int y_{[j}F_{i]}(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp} + \int y_{(j}F_{i)}(\boldsymbol{y}_{\rm sp})\,d^3\boldsymbol{y}_{\rm sp}\right] = 2\epsilon_{ijk}S_k\frac{n_j}{R^2} + 2\dot{I}_{ij}\frac{n_j}{R^2}. \tag{27}$$

Substituting in this, and its time derivative, gives

$$\bar{h}_{0i}(\boldsymbol{x}) = 2\epsilon_{ijk}S_k\frac{n_j}{R^2} + 2\dot{I}_{ij}\frac{n_j}{R^2} - 2\epsilon_{ijk}\dot{S}_k\frac{n_j}{R} - 2\ddot{I}_{ij}\frac{n_j}{R} + ...\Big|_{\rm ret}. \tag{28}$$

Using conservation of angular momentum, we drop the $\dot{S}_k$ term, leaving

$$\bar{h}_{0i}(\boldsymbol{x}) = 2\epsilon_{ijk}S_k\frac{n_j}{R^2} + 2\dot{I}_{ij}\frac{n_j}{R} - 2\ddot{I}_{ij}\frac{n_j}{R} + ...\Big|_{\rm ret}. \tag{29}$$

Finally, in the case of the space-space components we will write only the lowest term,

$$\bar{h}_{ij}(\boldsymbol{x}) = \frac{4}{R}\int T_{ij}\,d^3\boldsymbol{y}_{\rm sp}\Big|_{\rm ret}. \tag{30}$$

This can be simplified by noting that from Eq. (26):

$$\ddot{I}_{ij} = 2\int y_{(j}\dot{F}_{i)}\,d^3\boldsymbol{y}_{\rm sp} = -2\int y_{(j}T_{i)k,k}\,d^3\boldsymbol{y}_{\rm sp} = 2\int T_{(i}y_{j),k}\,d^3\boldsymbol{y}_{\rm sp} = 2\int T_{ij}\,d^3\boldsymbol{y}_{\rm sp}. \tag{31}$$

Then:

$$\bar{h}_{ij}(\boldsymbol{x}) = \frac{2}{R}\ddot{I}_{ij} + ...\Big|_{\rm ret}. \tag{32}$$

The far-field metric perturbation from a source according to the above equations can be found by reversing the trace of $\bar{h}_{ij}$; we need

$$\bar{h} = -4\frac{M}{R} + \frac{2}{R}\ddot{I}_{kk} - \frac{2}{R}n_in_j\ddot{I}_{ij} + 6\frac{Q_{ij}n_in_j}{R^3} + ...\Big|_{\rm ret}. \tag{33}$$

Including this with trace reversal gives

$$\begin{aligned} h_{00} &= 2\frac{M}{R} + \frac{1}{R}\ddot{I}_{kk} + \frac{1}{R}n_in_j\ddot{I}_{ij} + 3\frac{Q_{ij}n_in_j}{R^3} + ...\Big|_{\rm ret} \\ h_{0i} &= 2\epsilon_{ijk}S_k\frac{n_j}{R^2} + 2\dot{I}_{ij}\frac{n_j}{R^2} - 2\frac{n_j}{R}\ddot{I}_{ij} + ...\Big|_{\rm ret} \quad \text{and} \\ h_{ij} &= 2\frac{M}{R}\delta_{ij} + \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_kn_l\ddot{I}_{kl}\delta_{ij} - 3\frac{Q_{ij}n_in_j}{R^3} + ...\Big|_{\rm ret}. \end{aligned} \tag{34}$$

This looks at first glance to have several pieces: there is the familiar Newtonian potential; there is a piece associated with the angular momentum in the time-space part (*gravitomagnetism*); and there is a set of outward-propagating waves (gravitational waves!) associated with time variation of the quadrupole moment.

B. The radiation terms

You will note that the gravitational waves ($\dot{I}_{ij}$ terms) in the far field do not look like the ones we studied: in particular they have time-space and time-time components and are not orthogonal to n_i (the outward direction). At large distances from the source, the leading-order ($1/R$) wave terms are

$$\begin{aligned} h_{00}^{\rm GW} &= \frac{1}{R}\ddot{I}_{kk} + \frac{1}{R}n_in_j\ddot{I}_{ij}, \\ h_{0i}^{\rm GW} &= -2\frac{n_j}{R}\ddot{I}_{ij}, \quad \text{and} \\ h_{ij}^{\rm GW} &= \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_kn_l\ddot{I}_{kl}\delta_{ij} \end{aligned} \tag{35}$$

(all retarded). Fortunately, a gauge transformation can be used to eliminate most of these terms. Recall that under a change of coordinates ξ^μ, the metric tensor changed by $\Delta h_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu}$. We first perform a change of the time coordinate t,

$$\xi_0 = \frac{1}{2R}\dot{I}_{kk} + \frac{1}{2R}n_in_j\dot{I}_{ij}, \tag{36}$$

leaving the spatial coordinates fixed ($\xi_i = 0$). Then the time-time component changes by

$$\Delta h_{00} = -2\dot{\xi}_0 = -\frac{1}{R}\ddot{I}_{kk} - \frac{1}{R}n_in_j\ddot{I}_{ij}. \tag{37}$$

To find the correction $\Delta h_{0i} = -\xi_{0,i}$, we recall that since $\dot{I}_{ij}$ is evaluated at the retarded time, its spatial derivative contains a time derivative:

$$\partial_i I_{kl} = \dot{I}_{kl}\partial_i(t-R) = -\dot{I}_{kl}\partial_i R = -\dot{I}_{kl}n_i. \tag{38}$$

Thus:

$$\Delta h_{0i} = \frac{1}{2R}n_i\ddot{I}_{kk} + \frac{1}{2R}n_in_jn_k\ddot{I}_{jk}. \tag{39}$$

After applying this transformation, the outgoing wave is

$$\begin{aligned} h_{00}^{\rm GW} &= 0, \\ h_{0i}^{\rm GW} &= \frac{1}{2R}n_i\ddot{I}_{kk} + \frac{1}{2R}n_in_jn_k\ddot{I}_{jk} - 2\frac{n_j}{R}\ddot{I}_{ij}, \quad \text{and} \\ h_{ij}^{\rm GW} &= \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_kn_l\ddot{I}_{kl}\delta_{ij} \end{aligned} \tag{40}$$

(again retarded).

Finally, we may introduce a gauge transformation in the spatial coordinates to eliminate $h_{0i}^{\rm GW}$. We want $\Delta h_{0i} = -\dot{\xi}_i$ to cancel $h_{0i}^{\rm GW}$, so we choose ξ_i to be the time-integral of $h_{0i}^{\rm GW}$:

$$\xi_i = \frac{1}{2R}n_i\dot{I}_{kk} + \frac{1}{2R}n_in_jn_k\dot{I}_{jk} - 2\frac{n_j}{R}\dot{I}_{ij}. \tag{41}$$

Now we find that the change to the purely spatial metric is

$$\begin{aligned}\Delta h_{ij} &= -\xi_{i,j} - \xi_{j,i} \\ &= n_j \dot{\xi}_i + n_i \dot{\xi}_j \\ &= \frac{1}{R} n_i n_j \ddot{I}_{kk} + \frac{1}{R} n_i n_j n_k n_l \ddot{I}_{kl} - \frac{2}{R} n_j n_k \ddot{I}_{ik} - \frac{2}{R} n_i n_k \ddot{I}_{jk}. \end{aligned} \tag{42}$$

The overall amplitude of the outgoing gravitational wave is then

$$\begin{aligned} h^{\rm GW}_{00} &= 0, \\ h^{\rm GW}_{0i} &= 0, \quad \text{and} \\ h^{\rm GW}_{ij} &= \frac{1}{R}\left(2\ddot{I}_{ij} - \ddot{I}_{kk}\delta_{ij} + n_k n_l \ddot{I}_{kl}\delta_{ij} + n_i n_j \ddot{I}_{kk} + n_i n_j n_k n_l \ddot{I}_{kl} - 2 n_j n_k \ddot{I}_{ik} - 2 n_i n_k \ddot{I}_{jk}\right) \end{aligned} \tag{43}$$

(all retarded, as usual). This is the usual form for the amplitude of emitted gravitational waves.

It is straightforward to check that Eq. (43) corresponds to a transverse-traceless tensor. To see that it is transverse, note that

$$h^{\rm GW}_{ij} n_i = \frac{1}{R}\left(2\ddot{I}_{ij} n_i - \ddot{I}_{kk} n_j + n_i n_k n_l \ddot{I}_{kl} + n_j \ddot{I}_{kk} + n_j n_k n_l \ddot{I}_{kl} - 2 n_j n_k n_i \ddot{I}_{ik} - 2 n_k \ddot{I}_{jk}\right) = 0. \tag{44}$$

To see that it is traceless, take

$$h^{\rm GW}_{ii} = \frac{1}{R}\left(2\ddot{I}_{ii} - 3\ddot{I}_{kk} + 3 n_k n_l \ddot{I}_{kl} + \ddot{I}_{kk} + n_k n_l \ddot{I}_{kl} - 2 n_i n_k \ddot{I}_{ik} - 2 n_i n_k \ddot{I}_{ik}\right) = 0. \tag{45}$$

The equation for $h^{\rm GW}_{ij}$ may look messy (it has 7 terms) but in fact the first one is the key: the others are simply projections that guarantee the transverse-traceless nature.

C. The non-radiation terms

In addition to gravitational radiation, the metric can contain the non-radiation terms, the leading ones of which are associated with the mass M and angular momentum ${}^{(3)}\boldsymbol{S}$:

$$ds^2 = -\left(1 - 2\frac{M}{R}\right) dt^2 + 4\epsilon_{ijk} n_j \frac{S_k}{R^2} dt\, dx^i + \left(1 + 2\frac{M}{R}\right)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + ... \tag{46}$$

The "mass" terms M we have already discussed extensively: indeed, the mass M of a system is **measurable** using ordinary Newtonian dynamics, e.g. using Kepler's 3rd law.

The "angular momentum" terms ${}^{(3)}\boldsymbol{S}$ are a bit trickier: they lead to *gravitomagnetism*. A contribution h_{0i} to the time-space part of the metric in linear theory leads to a contribution to the Christoffel symbol:

$$\Delta \Gamma^i{}_{0j} = -\frac{1}{2} h_{0j,i} + \frac{1}{2} h_{0i,j} = h_{0[i,j]}, \tag{47}$$

so that there is an additional contribution to the second derivative of a coordinate:

$$\Delta \frac{d^2 x^i}{d\tau^2} = -2 h_{0[i,j]} \dot{x}^j \left(\frac{dt}{d\tau}\right)^2 \approx -2 h_{0[i,j]} \dot{x}^j. \tag{48}$$

Therefore if we define the 3-vector ${}^{(3)}\boldsymbol{A}$ (the gravitomagnetic 3-vector potential) with components h_{0i}, we see that an object has an apparent acceleration given by

$${}^{(3)}\boldsymbol{a}_{\rm gravitomagnetic} = {}^{(3)}\boldsymbol{v} \times (\nabla \times {}^{(3)}\boldsymbol{A}). \tag{49}$$

Thus the h_{0i} term, generated by the total angular momentum of an object, is exactly analogous in its low-velocity limit to the magnetic field interaction in electrodynamics. There is a difference though – it is common in both nature and in the laboratory for magnetic forces to dominate over electric forces. This is because charges of both signs exist; a current-carrying wire has equal numbers of protons and electrons, but there is a drift of the electrons relative to the protons (or nuclei). Thus the wire's electric field may be small and the magnetic forces become dominant. For gravitation, no such circumstance occurs: ρv in most systems is small compared with ρ.