Lecture IX: Weak field tests of GR: the gravitational redshift, deflection of light, and Shapiro delay

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I. OVERVIEW

General relativity is supposed to be a *theory* of nature – i.e. it makes predictions that must be compared to observation or experiment. We will cover here the tests for which we have developed the appropriate tools:

- The *gravitational redshift*: a photon in a static spacetime changes its energy in accordance with the gravitational potential, $\Delta\nu/\nu = -\Delta\Phi$.
- The *deflection of light* by the Sun.
- The *Shapiro delay*, or shift in radar ranging of an object due to the presence of a massive body along the line of sight.

A fourth classical test of GR – the perihelion advance of Mercury – requires one to go beyond linearized gravity so we will not treat it here.

The theory in MTW §40.1–40.4 is still valid. The description of experiments is however very much out of date! (It is still of historical interest.)

II. GRAVITATIONAL REDSHIFT

The first, and simplest, of our tests of GR is the gravitational redshift. The answer in this case could have been "guessed" without all of the GR machinery, but it is still important as the experimental results are in fundamental discord with flat-spacetime descriptions of gravity.

A. Theory

Consider a spacetime with a Newtonian gravitational potential,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{1}$$

with $|\Phi| \ll 1$ (terms of order Φ^2 are dropped). We will suppose that Φ depends only on the spatial coordinates and is time-independent.

The problem is very simple: a photon emitted from one point $\mathcal{P}$ propagates to another point $\mathcal{Q}$, and frequencies of emission and absorption are compared by stationary observers ($u^i = 0$, where $\boldsymbol{u}$ is the 4-velocity of the observer – not the photon!). One wishes to determine what is the relation between $\nu_{\rm em}$ and $\nu_{\rm abs}$.

General relativity makes this problem quite straightforward. We first consider the 4-momentum of a photon, $p^\mu = dx^\mu/d\lambda$, where λ is the affine parameter (not the wavelength!). We see from the geodesic equation that

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta = \frac{1}{2}g^{\mu\nu}(g_{\alpha\beta,\nu} - g_{\beta\nu,\alpha} - g_{\alpha\nu,\beta})p^\alpha p^\beta = \frac{1}{2}g^{\mu\nu}g_{\alpha\beta,\nu}p^\alpha p^\beta - g^{\mu\nu}g_{\beta\nu,\alpha}p^\alpha p^\beta. \tag{2}$$

It's possible to work the problem by solving for the geodesic connecting $\mathcal{P}$ and $\mathcal{Q}$, enforcing the null condition to make sure that the photon arrives at the correct time, and then transforming p^μ at both $\mathcal{P}$ and $\mathcal{Q}$ to the local Lorentz system of a stationary observer. However it is much easier to solve this problem by converting to p_μ:

$$\frac{dp_\sigma}{d\lambda} = \frac{d}{d\lambda}(g_{\sigma\mu}p^\mu) = \frac{dg_{\sigma\mu}}{d\lambda}p^\mu + g_{\sigma\mu}\frac{dp^\mu}{d\lambda} = g_{\sigma\mu,\nu}p^\nu p^\mu + \frac{1}{2}g_{\alpha\beta,\sigma}p^\alpha p^\beta - g_{\beta\sigma,\alpha}p^\alpha p^\beta. \tag{3}$$

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The first and last terms on the right-hand side cancel. Then since $g_{\alpha\beta}$ is stationary (it does not depend on $t = x^0$) it follows that

$$\frac{dp_0}{d\lambda} = 0. \tag{4}$$

Therefore we have

$$p_0(\mathcal{P}) = p_0(\mathcal{Q}). \tag{5}$$

We're not quite done yet, because what we want to know is the physical energy $E = \boldsymbol{p} \cdot \boldsymbol{u}$ seen by each observer. However, for stationary observers ($u^i = 0$), the condition of normalization of the 4-velocity $\boldsymbol{u} \cdot \boldsymbol{u} = -1$ tells us that

$$u^0 = \frac{1}{\sqrt{-g_{00}}} = \frac{1}{\sqrt{1 + 2\Phi}} = 1 - \Phi. \tag{6}$$

Thus we find that

$$\frac{\nu_{\rm abs}}{\nu_{\rm em}} = \frac{E(\mathcal{Q})}{E(\mathcal{P})} = \frac{p_\alpha u^\alpha(\mathcal{Q})}{p_\alpha u^\alpha(\mathcal{P})} = \frac{u^0(\mathcal{Q})}{u^0(\mathcal{P})} = \frac{1 - \Phi(\mathcal{Q})}{1 - \Phi(\mathcal{P})} = 1 - \Delta\Phi, \tag{7}$$

where

$$\Delta\Phi = \Phi(\mathcal{Q}) - \Phi(\mathcal{P}). \tag{8}$$

We thus conclude that a photon changes its frequency as it moves in a gravitational potential. The fractional change in frequency is simply the change in gravitational potential, with a minus sign: photons get redder if they go up, and bluer if they go down.

B. Experimental tests

The gravitational redshift represents a daunting experimental challenge. The gradient of gravitational potential at the Earth's surface is

$$g = 9.8\,{\rm m\,s^{-2}} = 1.1 \times 10^{-16} c^2\,{\rm m}^{-1}. \tag{9}$$

So over a 22 m laboratory-scale height difference, a photon experiences a frequency shift $\Delta\nu/\nu = 2.5 \times 10^{-15}$.

Despite this tiny value, the shift was successfully measured in the *Pound-Rebka experiment*. The idea was to find an extremely narrow spectral line that could be emitted and then whose absorption could be detected somewhere else. The chosen line was the 14 keV line of the ^{57}Fe nucleus, which is produced in emission following the β-decay of ^{57}Co and can be absorbed by any iron-bearing material. The use of a nuclear line implies that perturbations to the energy level structure from surrounding material are much smaller than for atomic or molecular lines (but they are still measurable). The half-life of the excited state is 98 ns, and the natural width of the line is

$$\frac{\Delta\nu_{\rm half\ width\ half\ max}}{\nu} = \frac{1}{4\pi\nu\tau} = 2.3 \times 10^{-13}. \tag{10}$$

A concern in this type of experiment is that in emitting a photon, the parent nucleus recoils to conserve momentum, and consequently the emitted photon has slightly less than the line energy. This energy shift is

$$\frac{\Delta E}{E} = -\frac{p^2_{\rm recoil}/2m_{\rm nuc}}{E} = -\frac{E}{2m_{\rm nuc}} = -\frac{14\ {\rm keV}}{2 \cdot 57 \cdot 938\ {\rm MeV}} = -1.3 \times 10^{-7}. \tag{11}$$

This is enough to completely ruin the experiment; in fact the resulting photons are so far redward of the line that they will not be absorbed.

But the choice of a low-energy line enabled Pound and Rebka to make use of the *Mossbauer effect*: the modification of the recoil energy when the nucleus is part of a crystal. When a nucleus in free space emits a photon, it always recoils and takes away the energy given by Eq. (11). However, when a nucleus in a crystal recoils, the recoil energy is associated with the emission of phonons. (Of course the crystal's translational degrees of freedom always pick up some momentum, but the crystal is usually so massive that its recoil energy $-E/2m_{\rm crystal}$ can be neglected.) Moreover, the recoil energy predicted by Eq. (11) is -19 milli eV, whereas the typical energy per phonon in a metal is the Debye temperature,

$$E_{\rm phonon} \sim kT_{\rm Debye} \sim 40\,{\rm milli\ eV} \tag{12}$$

for Fe. Therefore the number of phonons emitted per decay is of order unity. This is important because phonon emission is a stochastic process: sometimes the number of phonons emitted is zero, in which case the energy shift of Eq. (11) disappears! The same mechanism works in absorption.

With the help of the Mossbauer effect, Pound and Rebka had all the tools they needed to measure the gravitational redshift. A strong ^{57}Co source was placed on a high level in the Jefferson physics building at Harvard, and an ^{57}Fe absorber followed by an X-ray detector was placed on a lower level. By moving the absorber and creating a Doppler shift, the setup enabled Pound and Rebka to scan the line profile and find the velocity that cancelled the gravitational redshift. To reduce systematic errors (associated with e.g. the part-per-trillion shifts in nuclear energy levels associated with the chemical environment) the experiment was repeated upside down. This was compared to the GR prediction:

$$\Delta v = 2.5 \times 10^{-15} = 0.75\,\mu{\rm m\,s}^{-1}. \tag{13}$$

In the improved version of the experiment [Pound & Snider, Phys. Rev. 140:788 (1965)], it was found that

$$\frac{\rm shift(measured)}{\rm shift(GR)} = 0.9990 \pm 0.0076({\rm stat}) \pm 0.010({\rm sys}). \tag{14}$$

A somewhat larger gravitational redshift can be observed in astronomical systems. For example, the surface of the Sun is redshifted by

$$\frac{\Delta\nu}{\nu} = -\frac{M}{R} = -\frac{GM}{c^2R} = -2\times 10^{-6}. \tag{15}$$

This is large enough to be measured by the redshift of its more prominent spectral lines (e.g. the Na I $3s_{1/2} - 3p_{1/2,3/2}$ doublet at 5890 and 5896 Å). However the absolute accuracy of the measurement is far worse than what one could measure with nuclear lines.

III. DEFLECTION OF LIGHT

Our next problem concerns the deflection of light rays by a massive body such as the Sun.

A. Theory

Let's consider a light ray passing by an object of mass M at impact parameter b. We wish to know by what angle it is deflected. We take the metric tensor to be

$$ds^2 = -\left(1 - 2\frac{M}{r}\right)dt^2 + \left(1 + 2\gamma\frac{M}{r}\right)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{16}$$

Why the parameter γ? GR tells us its value: $\gamma = 1$. But if our goal is to **test** GR, then we should calculate the deflection of light for a more general metric. Therefore one introduces the *parameterized post-Newtonian* (PPN) parameters, of which γ is the first that we will meet, measures their value, and compares to $\gamma_{\rm GR} = 1$.

We take as our setup a light ray traveling along the 3-axis, i.e. far from the object we have $x^1 = b$, $x^2 = 0$, and $x^3 = t = E\lambda$ (where E is the photon energy and λ is the affine parameter). The 4-momentum is $p^\mu = (E, 0, 0, E)$. Then for $M = 0$ this trajectory is maintained and the photon is not deflected. In the more general case, the photon is deflected, and correspondingly p^1 picks up a contribution, given by

$$p^1{}_{\rm final} = p^1{}_{\rm init} - \int_{-\infty}^{\infty} \Gamma^1{}_{\mu\nu}[x^\alpha(\lambda)]\,p^\mu p^\nu\,d\lambda. \tag{17}$$

Here the initial $p^1{}_{\rm init} = 0$. We may evaluate the integral to first order M using the *Born approximation*: since $\Gamma^1{}_{\mu\nu}$ is already first order, we may compute p^μ and $x^\alpha(\lambda)$ on the unperturbed trajectory:

$$p^1{}_{\rm final} = -\int_{-\infty}^{\infty} (\Gamma^1{}_{00} + 2\Gamma^1{}_{03} + \Gamma^1{}_{33})\,E^2\,d\lambda. \tag{18}$$

Moreover, the affine parameter is related to x^3 via $dx^3/d\lambda = E$, so

$$p^1{}_{\rm final} = -E \int_{-\infty}^{\infty} (\Gamma^1{}_{00} + 2\Gamma^1{}_{03} + \Gamma^1{}_{33})\, dx^3. \tag{19}$$

Now we need only evaluate the Christoffel symbols and do the integral. They are:

$$\begin{aligned} \Gamma^1{}_{00} &= \frac{1}{2} g^{11}(-g_{00,1} + 2g_{01,0}) = -M\partial_1 \frac{1}{r} = M\frac{x^1}{r^3}, \\ \Gamma^1{}_{03} &= 0, \quad \text{and} \\ \Gamma^1{}_{33} &= \frac{1}{2} g^{11}(-g_{33,1} + 2g_{31,3}) = -M\partial_1 \frac{\gamma}{r} = \gamma M\frac{x^1}{r^3}. \end{aligned} \tag{20}$$

Therefore we have

$$p^1{}_{\rm final} = -(1+\gamma)ME \int_{-\infty}^{\infty} \frac{x^1}{r^3}\, dx^3, \tag{21}$$

where the integral is along the unperturbed trajectory: $x^1 = b$, $x^2 = 0$. Solving it gives a deflection angle

$$\begin{aligned} \theta_{\rm d} &= -\frac{p^1{}_{\rm final}}{E} \\ &= (1+\gamma)M \int_{-\infty}^{\infty} \frac{x^1}{r^3}\, dx^3 \\ &= (1+\gamma)Mb \int_{-\infty}^{\infty} \frac{1}{[b^2 + (x^3)^2]^{3/2}}\, dx^3. \end{aligned} \tag{22}$$

(The sign is chosen so that positive $\theta_{\rm d}$ corresponds to light deflected toward the object.) This integral is evaluated using the transformation $x^3 = b\tan\alpha$, leading to

$$\theta_{\rm d} = (1+\gamma)Mb \int_{-\pi/2}^{\pi/2} b^{-2} \cos\alpha\, d\alpha = 2(1+\gamma)\frac{M}{b}. \tag{23}$$

In particular, GR predicts

$$\theta_{\rm d} = 4\frac{M}{b}. \tag{24}$$

Of course, all of this is valid only for $M/b \ll 1$, i.e. gravitational potentials small compared to c^2. With the exception of neutron stars and black holes, this is a good approximation.

B. Experimental tests

The best candidate in our solar system for such a test is the Sun. Given a mass of 2×10^{33} g and a radius of 7×10^{10} cm, the angle of deflection of a light ray that just grazes the surface of the Sun should be

$$\theta_{\rm d} = \frac{4M_\odot}{R_\odot} = \frac{4GM_\odot}{c^2 R_\odot} = 8.5 \times 10^{-6}\,\text{radians} = 1.75\,\text{arcsec}. \tag{25}$$

Thus if one looks at the background stars when the Sun is near them, they should appear shifted slightly farther away from the Sun than when the Sun is absent. (Yes, farther away: if light bends toward the Sun then the apparent position is shifted away.)

Clearly looking at the stars behind the Sun represents a major technical challenge. It has been solved in two ways. The first was the measurement during total solar eclipses. For many decades, measurements at remote locations in the paths of total eclipses led to measurements of the deflection with accuracies of a few tens of percents. The best measurements based on optical starlight were ultimately performed by the *Hipparcos* astrometric satellite [Froeschle, Mignard, & Arenou, Proc. ESA Symp. 402:49 (1997)]:

$$\gamma = 0.997 \pm 0.003. \tag{26}$$

Most noteworthy about this measurement is that Hipparcos never looked toward the Sun: measurements were done at angles of 47–133° from the Sun, and the corresponding deflection angles were no larger than 10 mas!

In the 1960s it became possible to measure positions of celestial sources very precisely, even in daytime, using radio interferometers. In particular, the brightest microwave point source in the sky, the blazar 3C279 ($z = 0.54$), lies only 12 arcmin from the plane of Earth's orbit and is eclipsed by the Sun once each year (on October 8). The first successful such measurements were carried out in October 1969, one at Owens Valley [Seielstad, Sramek, & Weiler, PRL 24:1373 (1970)] and one at Goldstone [Muhleman, Ekers, Fomalont, PRL 24:1377 (1970)]:

$$\gamma = \begin{cases} 1.02 \pm 0.23 & \text{Owens Valley} \\ 1.08^{+0.30}_{-0.20} & \text{Goldstone} \end{cases} . \tag{27}$$

(Remember that $\gamma = -1$ corresponds to no deflection.) A recent study of 3C279 using the continent-scale Very Long Baseline Array [Fomalont et al, ApJ 699:1395 (2009)] gives

$$\gamma = 0.9998 \pm 0.0003. \tag{28}$$

The limiting aspect of the radio measurements has been plasma-induced refraction effects as the radio waves propagate through the Sun's corona. Observations at multiple frequencies, and the access to higher frequencies with modern technology, can improve this because the refraction effects scale as $\propto \nu^{-2}$.

IV. SHAPIRO DELAY

Our third test is the *Shapiro delay* – the change in round trip light travel time to an object resulting from the influence of a massive body.

A. Theory

The setup for the Shapiro delay problem involves two stations, $\mathcal{A}$ and $\mathcal{B}$, at large distances from the source. A radio signal is emitted from $\mathcal{A}$ and travels to $\mathcal{B}$. Upon receiving the signal, $\mathcal{B}$ transmits a signal back to $\mathcal{A}$. The observer $\mathcal{A}$ measures the total proper time elapsed during the process. For simplicity, we will treat $\mathcal{A}$ and $\mathcal{B}$ as stationary relative to the star, and suppose that the impact parameter is b. The spatial coordinates of the observers are $(b, 0, x^3_{\mathcal{A}})$ and $(b, 0, x^3_{\mathcal{B}})$, and we will take $x^3_{\mathcal{A}} < 0$, $x^3_{\mathcal{B}} > 0$, and $|x^3_{\mathcal{A}}|, |x^3_{\mathcal{B}}| \gg b$.

The trajectory of the radio signal from $\mathcal{A}$ to $\mathcal{B}$ is a null trajectory, i.e. $g_{\mu\nu}p^\mu p^\nu = 0$, and therefore

$$-\left(1 - 2\frac{M}{r}\right) dt^2 + \left(1 + 2\gamma\frac{M}{r}\right) [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] = 0. \tag{29}$$

The unperturbed trajectory has x^1 and x^2 constant, so to first order in M we may neglect $(dx^1)^2$ and $(dx^2)^2$ in the above equations. Then it follows that

$$\left|\frac{dx^3}{dt}\right| = \sqrt{\frac{1 - 2M/r}{1 + 2\gamma M/r}} = 1 - (1 + \gamma)\frac{M}{r}. \tag{30}$$

The coordinate time required for radio waves to travel from $\mathcal{A}$ to $\mathcal{B}$ is

$$\begin{aligned} \Delta t_{\mathcal{A}\to\mathcal{B}} &= \int_{x^3_{\mathcal{A}}}^{x^3_{\mathcal{B}}} \frac{dt}{dx^3}\, dx^3 \\ &= \int_{x^3_{\mathcal{A}}}^{x^3_{\mathcal{B}}} \left[1 + (1 + \gamma)\frac{M}{r}\right] dx^3 \\ &= x^3_{\mathcal{B}} - x^3_{\mathcal{A}} + (1 + \gamma)M \int_{x^3_{\mathcal{A}}}^{x^3_{\mathcal{B}}} \frac{1}{r}\, dx^3. \end{aligned} \tag{31}$$

Again we set $x^3 = b\tan\alpha$, and find that

$$\Delta t_{\mathcal{A}\to\mathcal{B}} = x^3_{\mathcal{B}} - x^3_{\mathcal{A}} + (1 + \gamma)M \int_{\mathcal{A}}^{\mathcal{B}} \sec\alpha\, d\alpha$$

$$= x^3_{\mathcal{B}} - x^3_{\mathcal{A}} + (1+\gamma)M \ln(\tan\alpha + \sec\alpha)|^{\mathcal{B}}_{\mathcal{A}}$$
$$= x^3_{\mathcal{B}} - x^3_{\mathcal{A}} + (1+\gamma)M \ln\left(\frac{x^3}{b} + \sqrt{1+\frac{(x^3)^2}{b^2}}\right)\Bigg|^{\mathcal{B}}_{\mathcal{A}}. \tag{32}$$

Now we must consider the limits of integration carefully, since as $x^3 \to \pm\infty$, the relativistic correction – the second term in Eq. (32) – becomes infinite! This is a consequence of the long-range nature of gravity ($\int dr/r$ is divergent). Similar "infrared divergences" occur in the study of other long-range forces, e.g. in QED. Fortunately, they are usually problems with the mathematical limit being taken rather than the theory itself, and the Shapiro delay is no exception. At large positive x^3, we find

$$\ln\left(\frac{x^3}{b} + \sqrt{1+\frac{(x^3)^2}{b^2}}\right) \to \ln\frac{2x^3}{b}, \tag{33}$$

whereas at large negative x^3 we find by Taylor expansion of the square root

$$\ln\left(\frac{x^3}{b} + \sqrt{1+\frac{(x^3)^2}{b^2}}\right) \to \ln\left[-\frac{|x^3|}{b} + \frac{|x^3|}{b} + \frac{1}{2|x^3|/b}\right] = -\ln\frac{2|x^3|}{b}. \tag{34}$$

Theerfore, Eq. (32) simplifies in the limit of $\mathcal{A}$ and $\mathcal{B}$ much farther from the Sun than b:

$$\Delta t_{\mathcal{A}\to\mathcal{B}} = x^3_{\mathcal{B}} - x^3_{\mathcal{A}} + (1+\gamma)M\left(\ln\frac{2x^3_{\mathcal{B}}}{b} + \ln\frac{2x^3_{\mathcal{A}}}{b}\right). \tag{35}$$

This is the coordinate time; the time observed by $\mathcal{A}$ should have an additional factor of

$$\frac{d\tau}{dt} = \sqrt{-g_{00}} = 1 + \frac{M}{|x^3_{\mathcal{A}}|}, \tag{36}$$

and to get a round trip time we should double our result. Thus the total proper time observed by $\mathcal{A}$ before the return of the signal is

$$\tau_{\mathcal{A}\to\mathcal{B}\to\mathcal{A}} = 2\left(1+\frac{M}{r_{\mathcal{A}}}\right)(r_{\mathcal{A}} + r_{\mathcal{B}}) + 2(1+\gamma)M\ln\frac{4r_{\mathcal{A}}r_{\mathcal{B}}}{b^2}. \tag{37}$$

Now of course we can't test GR simply by measuring the absolute round-trip time to an object; how would we distinguish a 1 μs delay from the object simply being 150 m farther away? But we can watch the variation of the round-trip time as an object passes behind the Sun. As the impact parameter varies, one should see a round-trip time delay that varies as

$$\tau_{\mathcal{A}\to\mathcal{B}\to\mathcal{A}} = \text{constant} - 4(1+\gamma)M\ln b. \tag{38}$$

The $-$ sign here means that there is an increased delay for small impact parameters.

B. Experimental tests

The Shapiro delay was measured in the 1960s, first with passive radar measurements of the inner planets, and subsequently with active ranging experiments on interplanetary spacecraft.

A major advance was made in using an active transmitter on a spacecraft stationed on a planet (and thus free of the perturbing forces on most space missions). Using the *Viking* Mars landers, it was found [Reasenberg et al., ApJ 234:L219 (1979)] that

$$\gamma = 1.000 \pm 0.002. \tag{39}$$

The most precise measurement of the Shapiro delay from spacecraft measurements thus far has been from the *Cassini* mission during its cruise to Saturn [Bertotti, Iess & Tortora, Nature 425:374 (2003)]. Using ranging at multiple frequencies to calibrate out plasma time delay effects in the solar corona, it was found that

$$\gamma = 1.000021 \pm 0.000023. \tag{40}$$