Lecture VIII: Linearized gravity

Christopher M. Hirata
Caltech M/C 350-17, Pasadena CA 91125, USA[*]
(Dated: November 5, 2012)

I. OVERVIEW

We are now ready to consider the solutions of GR for the case of weak gravitational fields. This encompasses Newtonian gravity, as well as a few extensions thereof. In addition to its importance in e.g. the solar system, our study of linearized gravity will prepare us for the discussion of some of the weak-field experimental tests of GR (Lecture XI). It will also illustrate some of the issues that arise in solving the field equations.

The recommended reading for this lecture is:

- MTW Ch. 18.

II. LINEARIZATION OF EINSTEIN'S EQUATIONS

[Reading: MTW §18.1]

Linearized gravity is simply perturbation theory around Minkowski spacetime. That is, we assume that the metric tensor is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $|h_{\mu\nu}| \ll 1$ is a small perturbation. We will endeavor to construct the Einstein tensor to linear order in $h_{\mu\nu}$ and then use it to construct the linearized equation relating the metric perturbations to the stress-energy tensor.

For the purposes of linearized gravity analyses, we will raise and lower the indices on $h_{\mu\nu}$ according to η:

$$h^\mu{}_\nu \equiv \eta^{\mu\alpha} h_{\alpha\nu} \quad \text{and} \quad h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}. \tag{2}$$

In linear perturbation theory this doesn't matter, since if we had chosen instead to use the full metric $g^{\mu\alpha}$ the corrections would be second order. However it is standard practice in higher-order perturbation theory in GR to raise and lower indices of the metric perturbation according to the background – i.e. to think of the metric perturbation is a tensor field propagating on the background spacetime.

A. The Christoffel symbols and curvature tensors

It is straightforward to compute the Christoffel symbols in the linearized metric. They are

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(-h_{\alpha\beta,\nu} + h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha}) = \frac{1}{2}(-h_{\alpha\beta}{}^{,\mu} + h^\mu{}_{\alpha,\beta} + h^\mu{}_{\beta,\alpha}), \tag{3}$$

where in the second equality we used linearization. Note also that to linear order the covariant derivative of a metric perturbation can be transformed into a partial derivative.

One may then proceed to compute the Ricci tensor,

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} - \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\beta\nu} + \Gamma^\beta{}_{\mu\nu}\Gamma^\alpha{}_{\beta\alpha}; \tag{4}$$

[*]Electronic address: `chirata@tapir.caltech.edu`

to linear order the "ΓΓ" terms go away and we are left with

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} \\ &= \frac{1}{2}(-h_{\mu\nu,\alpha}{}^{\alpha} + h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h^{\alpha}{}_{\mu,\alpha\nu} - h^{\alpha}{}_{\alpha,\mu\nu}) \\ &= \frac{1}{2}(-h_{\mu\nu,\alpha}{}^{\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h^{\alpha}{}_{\alpha,\mu\nu}). \end{aligned} \tag{5}$$

It's convenient to simplify this with the notations

$$h \equiv h^{\alpha}{}_{\alpha} \quad \text{and} \quad \Box = \partial^{\alpha}\partial_{\alpha} = -\partial_t^2 + \nabla^2 \tag{6}$$

(the $\Box$ operator is called the *d'Alembertian* and is the 4D generalization of the Laplacian). Then

$$R_{\mu\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h_{,\mu\nu}). \tag{7}$$

The Ricci scalar is the trace of this,

$$R = R^{\mu}{}_{\mu} = \frac{1}{2}(-\Box h + h^{\mu\alpha}{}_{,\mu\alpha} + h^{\mu\alpha}{}_{,\mu\alpha} - \Box h) = h^{\mu\alpha}{}_{,\mu\alpha} - \Box h. \tag{8}$$

Finally, the Einstein tensor is to linear order

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h_{,\mu\nu} - h^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} + \eta_{\mu\nu}\Box h). \tag{9}$$

We thus arrive at the equation of linearized gravity:

$$-\Box h_{\mu\nu} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h_{,\mu\nu} - h^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} + \eta_{\mu\nu}\Box h = 16\pi T_{\mu\nu}. \tag{10}$$

B. Trace-reversed perturbation variable

Equation (10) is quite complicated. It is usually preferred to write the metric perturbation using the *trace-reversed perturbation variable*, defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \quad \leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}. \tag{11}$$

The normal metric perturbation $h_{\mu\nu}$ and the trace-reversed perturbation $\bar{h}_{\mu\nu}$ contain exactly the same information. (Note that we have used $\bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -h$, and assumed $n = 4$ dimensions.) Substituting into Eq. (10) gives

$$-\Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} + \bar{h}_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \frac{1}{2}\bar{h}_{,\mu\nu} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} - \frac{1}{2}\bar{h}_{,\mu\nu} + \bar{h}_{,\mu\nu} - \bar{h}^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} - \eta_{\mu\nu}\Box\bar{h} = 16\pi T_{\mu\nu}. \tag{12}$$

This simplifies to an equation with only 4 instead of 6 terms on the left-hand side:

$$-\Box\bar{h}_{\mu\nu} + \bar{h}_{\nu}{}^{\alpha}{}_{,\mu\alpha} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} - \bar{h}^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} = 16\pi T_{\mu\nu}. \tag{13}$$

C. Gauge ambiguities

We now have, in the limit of weak gravity, a system of 10 linear PDEs for 10 variables $\bar{h}_{\mu\nu}$ in terms of the sources (the stress-energy tensor). It is thus tempting to try to solve for $\bar{h}_{\mu\nu}$. This is, unfortunately, not possible because we have not chosen a coordinate system. Indeed, small deviations from $g_{\mu\nu} = \eta_{\mu\nu}$ may arise either because spacetime is perturbed from Minkowski, or because we perturbed the coordinate system (or both). So we must understand the implications of perturbing the coordinate system, or making a small *gauge transformation*.

To begin with, let's imagine two coordinate systems x^{μ} and $x^{\mu'}$, deviating from each other by a very small amount ξ^{μ}:

$$x^{\mu} = x^{\mu'} - \xi^{\mu}, \tag{14}$$

Then to first order in $\boldsymbol{\xi}$ the metric in the two systems is related by

$$\begin{aligned} g_{\mu'\nu'}(x') &= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta}(x) \\ &= (\delta^\alpha_{\mu'} - \xi^\alpha{}_{,\mu'})(\delta^\beta_{\nu'} - \xi^\beta{}_{,\nu'})[g_{\alpha\beta}(x') - g_{\alpha\beta,\sigma}(x')\xi^\sigma] \\ &= g_{\mu\nu}(x) - \xi^\alpha{}_{,\mu} g_{\alpha\nu} - \xi^\beta{}_{,\nu} g_{\mu\beta} - g_{\mu\nu,\sigma}\xi^\sigma. \end{aligned} \tag{15}$$

This is for a general infinitesimal change of coordinates. In the special case of Minkowski spacetime, we can neglect the last term (it is second order in perturbation theory) and we have

$$g_{\mu'\nu'}(x') - g_{\mu\nu}(x) = -\xi_{\nu,\mu} - \xi_{\mu,\nu}. \tag{16}$$

Thus an infinitesimal change of coordinates in which the "grid" is displaced by the vector $\boldsymbol{\xi}$ changes the metric perturbation according to

$$\Delta h_{\mu\nu} = -\xi_{\nu,\mu} - \xi_{\mu,\nu} \tag{17}$$

or

$$\Delta \bar{h}_{\mu\nu} = -\xi_{\nu,\mu} - \xi_{\mu,\nu} + \xi^\alpha{}_{,\alpha}\eta_{\mu\nu}. \tag{18}$$

In solving Einstein's equations it is common practice to impose *gauge conditions*: one adds new conditions on the metric tensor until the coordinate system is uniquely fixed. In doing so, one must prove that the gauge condition can be satisfied by some appropriate choice of $\boldsymbol{\xi}$. After 4 gauge conditions are imposed (the number of degrees of freedom in choosing the coordinate system), the metric is determined.

D. Lorentz gauge

The gauge in which linearized gravity is simplest is the *Lorentz gauge*, described by the choice

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{19}$$

Before we use Lorentz gauge we must prove that it exists, i.e. that we can always find a $\boldsymbol{\xi}$ that turns an arbitrary perturbed Minkowski space into one satisfying Eq. (19). To do so, we recall that a gauge transformation leads to a new trace-reversed perturbation,

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \xi^\alpha{}_{,\alpha}\eta_{\mu\nu}. \tag{20}$$

Taking the divergence gives

$$\bar{h}'^{\mu\alpha}{}_{,\alpha} = \bar{h}^{\mu\alpha}{}_{,\alpha} - \xi^{\alpha,\mu}{}_{\alpha} - \xi^{\mu,\alpha}{}_{\alpha} + \xi^\beta{}_{,\beta\alpha}\eta^{\mu\alpha} = \bar{h}^{\mu\alpha}{}_{,\alpha} - \Box\xi^\mu. \tag{21}$$

Therefore the transformed system is in Lorentz gauge if we can choose $\Box\xi^\mu = \bar{h}^{\mu\alpha}{}_{,\alpha}$. Fortunately, for any function f there is always a function F such that $\Box F = f$. Thus there is always a choice of gauge satisfying the Lorentz condition, Eq. (19). In fact there are many such functions, so the Lorentz gauge is not uniquely determined – we may impose any additional transformation by $\boldsymbol{\xi}$ with $\Box\boldsymbol{\xi} = 0$ and we are still in Lorentz gauge. We will come back to this issue later.

In Lorentz gauge, the linearized Einstein equation Eq. (13) becomes simply

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{22}$$

III. GRAVITATIONAL FIELD OF NONRELATIVISTIC MATTER

As a first application of linearized gravity, let's consider the gravitational field of an isolated distribution of masses in the Newtonian limit. That is, we assume that (i) the stress-energy tensor is dominated by the density, $T_{\mu\nu} = \rho\delta_{\mu 0}\delta_{\nu 0}$; (ii) the matter moves slowly enough that we can neglect time derivatives in the equations; and (iii) the spacetime is "asymptotically flat," i.e. at large distances approaches the behavior of Minkowski spacetime. Then Eq. (22) says that

$$\nabla^2\bar{h}_{00} = -16\pi\rho, \tag{23}$$

with all other components of $\bar{h}_{\mu\nu}$ equal to zero. Let us define the *Newtonian potential* Φ by the equation

$$\nabla^2\Phi = 4\pi\rho, \tag{24}$$

subject to the usual boundary condition that $\Phi \to 0$ at infinite distance. Then we have

$$\bar{h}_{00} = -4\Phi. \tag{25}$$

We may solve for the entire metric perturbation by noting that $\bar{h} = -\bar{h}_{00} = 4\Phi$ and hence

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu} = \begin{cases} -2\Phi & \mu = \nu = 0 \\ -2\Phi & \mu = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{26}$$

Thus the line element is

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{27}$$

This is our first "solution" (albeit only in perturbation theory) of Einstein's equation!

A case of particular interest is the field generated by a compact massive body of mass M. The Newtonian potential is now $\Phi = -M/r$, where $r = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$. The line element is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{28}$$

This is, to first order in M, the description of the spacetime around the Sun. We will use it in the next lecture to discuss the gravitational deflection of light and the Shapiro time delay.

IV. GRAVITATIONAL WAVES

[Reading: MTW §18.2]

Let us now leave the realm of nearly static spacetimes and explore the full dynamics of linearized gravity. We will start by investigating the vacuum solutions of Einstein's equations, and discover that they allow wave motions to propagate in the structure of spacetime. In later lectures we will investigate the generation of gravitational waves by matter.

In vacuum, the Lorentz gauge equations become $\Box\bar{h}_{\mu\nu} = 0$. This is a system of 10 simple scalar wave equations, and we can search for solutions of the form

$$\bar{h}_{\mu\nu} = \Re\left(A_{\mu\nu}e^{ik_\alpha x^\alpha}\right) = \Re\left(A_{\mu\nu}e^{ik_i x^i}e^{-i\omega t}\right), \tag{29}$$

where k_μ is the wave vector and $\omega = k^0 = -k_0$. The d'Alembertian acting on a complex exponential is

$$\Box = \partial_\alpha\partial^\alpha = (ik_\alpha)(ik^\alpha) = -k_\alpha k^\alpha, \tag{30}$$

so we have a solution if $\boldsymbol{k}$ is a null vector, i.e. $k_\alpha k^\alpha = 0$ or

$$\omega = \pm\sqrt{(k_1)^2 + (k_2)^2 + (k_3)^2}. \tag{31}$$

From the dispersion relation we can see that all perturbations will have both phase and group velocities equal to the speed of light.

It looks at first glance like there are 10 polarizations of gravitational wave since there are 10 scalar wave equations. However, this is not the case. The Lorentz gauge condition $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ tells us that

$$A^{\mu\alpha}k_\alpha = 0. \tag{32}$$

This restriction eliminates 4 of the degrees of freedom, so there are only 6 legal degrees of freedom in $A_{\mu\nu}$. Moreover, there is a residual gauge freedom in the Lorentz gauge. If we introduce a gauge transformation $\boldsymbol{\xi}$ given by

$$\xi^\mu = \Re\left(iB^\mu e^{ik_\alpha x^\alpha}\right), \tag{33}$$

then since $\Box\xi^\mu = 0$ the Lorentz gauge condition is preserved. It follows that the metric perturbation changes by

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \xi^\alpha{}_{,\alpha}\eta_{\mu\nu} \tag{34}$$

and the gravitational wave amplitude changes by

$$A_{\mu\nu} \to A_{\mu\nu} + k_\mu B_\nu + k_\nu B_\mu - k_\alpha B^\alpha \eta_{\mu\nu}. \tag{35}$$

We may choose the 4 components of B_μ however we want. It is most convenient to look at the 4 quantities:

$$\begin{pmatrix} \frac{1}{2}A^\alpha{}_\alpha \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}A^\alpha{}_\alpha \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix} + \begin{pmatrix} -\omega & -k_1 & -k_2 & -k_3 \\ k_1 & -\omega & 0 & 0 \\ k_2 & 0 & -\omega & 0 \\ k_3 & 0 & 0 & -\omega \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{36}$$

Since the 4×4 matrix shown is invertible, one can always choose the coordinate system so that

$$A^\alpha{}_\alpha = 0 \quad \text{and} \quad A_{0i} = 0. \tag{37}$$

This choice, combined with Eq. (32), is called *transverse-traceless gauge* for reasons that will become clear shortly.

This implies a total of 8 conditions on $A_{\mu\nu}$ so there are 2 remaining polarizations of *gravitational waves*. Also the inversion completely fixes the gauge (i.e. it specifies B_μ) so these waves are "real" in the sense that a gauge transformation cannot eliminate them.

What is the nature of these two polarizations? We may find out by expanding Eq. (32), the $\mu = 0$ component of which gives

$$-\omega A^{00} + k_i A^{0i} = 0. \tag{38}$$

Then Eq. (37) tells us that for $\omega \neq 0$ we have $A^{00} = 0$ and hence from Eq. (37) $A_{ii} = A^\alpha{}_\alpha + A^{00} = 0$. Moreover, the spatial components of Eq. (32) give

$$A_{ij}k_j = 0. \tag{39}$$

Thus we see that $A_{\mu\nu}$ is (i) purely spatial (00 and $0i$ components vanish); (ii) traceless, $A_{ii} = 0$; and (iii) transverse, $A_{ij}k_j = 0$.

An explicit construction is possible if we consider a wave propagating in the 3-direction. Then the restrictions on $\mathbf{A}$ give

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

We note that under gauge transformations, Eq. (35), the quantities $A^{\rm g}_+ = \frac{1}{2}(A_{11} - A_{22})$ and $A^{\rm g}_\times = A_{12}$ do not change, so one may use these as a way to compute the amplitudes of an arbitrary gravitational wave. The metric associated with a gravitational wave propagating in the 3-direction is

$$ds^2 = -dt^2 + [1 + \Re(A_+ e^{ik_\alpha x^\alpha})](dx^1)^2 + 2\Re(A_\times e^{ik_\alpha x^\alpha})\,dx^1dx^2 + [1 - \Re(A_+ e^{ik_\alpha x^\alpha})](dx^2)^2 + (dx^3)^2. \tag{41}$$

What is the physical interpretation of this wave? Let us first consider a *linearly polarized* wave with + polarization, and consider a suite of test particles at constant spatial coordinates (x^1, x^2, x^3). [Exercise: prove that these particles are freely falling.] As the wave passes by, the particles separated in the 1-direction see each other move closer together and farther apart. The particles separated in the 2-direction see the same thing, except that the stretching and squeezing in the 2-direction is 180° out of phase with the 1-direction. The distance element in the 3-direction (the direction of propagation) is unaffected by the passage of the gravitational wave.

A $\times$ polarized wave yields the same phenomenon, except that the stretching and squeezing occurs along the axes $(2^{-1/2}, 2^{-1/2}, 0)$ and $(-2^{-1/2}, 2^{-1/2}, 0)$. Once again, the distance element in the 3-direction is unaffected. This is the reason for the designations of "+" and "×" polarization rather than "horizontal" and "vertical" as for light.

Just as for electromagnetic radiation, *circularly polarized* gravitational waves may be generated by superposing linearly polarized waves of equal amplitude with a 90° phase shift. For example,

$$ds^2 = -dt^2 + [1 + A_{\rm R}\cos(kx^3 - \omega t)](dx^1)^2 + 2A_{\rm R}\sin(kx^3 - \omega t)\,dx^1dx^2 + [1 - A_{\rm R}\cos(kx^3 - \omega t)](dx^2)^2 + (dx^3)^2. \tag{42}$$

The axis of stretching of this wave rotates through 180° every wave period. It thus has a *pattern speed* of $\frac{1}{2}\omega$. In the most general case, gravitational waves can be *elliptically polarized*.

An alternative and frequently useful description of gravitational waves involves the Weyl tensor, $C_{\alpha\beta\gamma\delta}$. For a gravitational wave in vacuum, this is the same as the Riemann tensor; straightforward computation shows that in the Lorentz gauge:

$$C_{\alpha\beta\gamma\delta} = 2k_{[\alpha}\bar{h}_{\beta][\delta}k_{\gamma]}. \tag{43}$$

V. CONSERVATION OF ENERGY AND MOMENTUM IN LINEARIZED GRAVITY

[Reading: MTW §18.3]

We finally return to our starting point in developing Einstein's equations: the conservation law for energy-momentum. The Lorentz gauge condition and the field equation tell us that

$$T^{\mu\nu}{}_{,\nu} = -\frac{1}{16\pi}\Box\bar{h}^{\mu\nu}{}_{,\nu} = 0. \tag{44}$$

This is of course the lowest approximation to the true conservation law $T^{\mu\nu}{}_{;\nu} = 0$. However it is paradoxical: it says, for example, that energy and momentum are locally conserved. Thus if one puts a mass in the Earth's gravitational field, Eq. (44) tells us that its momentum is conserved, i.e. that

$$P^i = \int_{\rm object} T^{0i}\, d^3x \tag{45}$$

has a time derivative given by boundary terms:

$$\dot{P}^i = \int_{\rm object} T^{0i}{}_{,0}\, d^3x = -\int_{\rm object} T^{ji}{}_{,j}\, d^3x = -\oint_{\rm boundary} T^{ji} n_j\, d^2x = 0 \tag{46}$$

if the region outside the object is vacuum. So if started at rest it does not fall! Therefore, **the linearized theory of gravity is not self-consistent**.

This apparent contradiction is resolved only in the full nonlinear theory of Einstein gravity, in which the conservation law is $T^{\mu\nu}{}_{;\nu} = 0$. The treatment of this resolution beyond linear perturbation theory will also clear up some other issues: the meaning of "gravitational energy" (either static or the energy of gravitational waves), the "total mass" and "total angular momentum" of a relativistic system, and the "conservation of energy" (how does a system emitting gravitational waves lose mass if energy-momentum is locally conserved?).