Lecture III: Tensor calculus and electrodynamics in flat spacetime

Christopher M. Hirata
Caltech M/C 350-17, Pasadena CA 91125, USA[*]
(Dated: October 8, 2012)

I. OVERVIEW

In this lecture we will continue to develop tensor calculus and apply it to the formulation of Maxwell's equations in flat spacetime.

The recommended reading for this lecture is:

- MTW Ch. 3.

II. MORE TENSOR OPERATIONS

Last time we defined general tensors as linear machines, and described how to raise and lower their indices (i.e. change whether they accepted vectors or 1-forms as inputs). But so far we haven't defined general tensor operations.

A. Outer product

Given two tensors $\mathbf{A}$ and $\mathbf{B}$ of rank $\binom{M}{N}$ and $\binom{P}{Q}$, we can define the *outer product* (or "tensor product") to be the tensor $\mathbf{A} \otimes \mathbf{B}$ with

$$(\mathbf{A}\otimes\mathbf{B})(\tilde{\boldsymbol{k}}, ...\boldsymbol{u}, \tilde{\boldsymbol{l}}, ...\boldsymbol{v}) = \mathbf{A}(\tilde{\boldsymbol{k}}, ...\boldsymbol{u})\mathbf{B}(\tilde{\boldsymbol{l}}, ...\boldsymbol{v}). \tag{1}$$

It is trivial to see that this is in fact a tensor – it is linear in each input. Its components are

$$(\mathbf{A}\otimes\mathbf{B})^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N}{}^{\gamma_1...\gamma_P}{}_{\delta_1...\delta_Q} = A^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N} B^{\gamma_1...\gamma_P}{}_{\delta_1...\delta_Q}. \tag{2}$$

This is the generalization to tensors of arbitrary rank of taking two column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and making a matrix $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$.

B. Gradient of a tensor

We have already defined the gradient of a scalar f, which is a 1-form $\boldsymbol{d}f$. More explicitly, the gradient of a scalar is a linear machine that takes a vector – a "velocity" $\boldsymbol{v} = d\mathcal{P}/d\lambda$, located at $\mathcal{P}(\lambda = 0)$ – and returns the scalar:

$$\boldsymbol{d}f(\boldsymbol{v}) = \frac{d}{d\lambda} f[\mathcal{P}(\lambda)]\bigg|_{\lambda=0} = \lim_{\epsilon\rightarrow 0} \frac{f[\mathcal{P}(\epsilon)] - f[\mathcal{P}(0)]}{\epsilon}. \tag{3}$$

Its components are the partial derivatives,

$$(\boldsymbol{d}f)_\alpha = \frac{\partial f}{\partial x^\alpha}. \tag{4}$$

Note that the rank of the gradient is 1 more than the rank of the object on which it acts. In this case, a scalar, formally a tensor of rank $\binom{0}{0}$, is turned into a tensor of rank $\binom{0}{1}$.

---

[*]Electronic address: `chirata@tapir.caltech.edu`

We can analogously define the gradient of a tensor $\mathbf{S}$ of any rank. For example, if $\mathbf{S}$ has rank $\binom{1}{1}$ (components $S^\alpha{}_\beta$), we write

$$\boldsymbol{\nabla}\mathbf{S}(\tilde{\boldsymbol{k}}, \boldsymbol{u}, \boldsymbol{v}) = \left.\frac{d}{d\lambda}\mathbf{S}[\mathcal{P}(\lambda)](\tilde{\boldsymbol{k}}, \boldsymbol{u})\right|_{\lambda=0} = \lim_{\epsilon\to 0}\frac{\mathbf{S}[\mathcal{P}(\epsilon)](\tilde{\boldsymbol{k}}, \boldsymbol{u}) - \mathbf{S}[\mathcal{P}(0)](\tilde{\boldsymbol{k}}, \boldsymbol{u})}{\epsilon}, \tag{5}$$

where $\tilde{\boldsymbol{k}}$ is a constant 1-form and $\boldsymbol{u}$ is a constant vector. Again there is one new slot: $\mathbf{S}$ had rank $\binom{1}{1}$ and $\boldsymbol{\nabla}\mathbf{S}$ has rank $\binom{1}{2}$. Using the chain rule, we can write this in component form as

$$(\boldsymbol{\nabla}\mathbf{S})^\alpha{}_{\beta\gamma}k_\alpha u^\beta v^\gamma = \frac{d}{d\lambda}(S^\alpha{}_\beta k_\alpha u^\beta) = v^\gamma\frac{\partial S^\alpha{}_\beta}{\partial x^\gamma}k_\alpha u^\beta = \frac{\partial S^\alpha{}_\beta}{\partial x^\gamma}k_\alpha u^\beta v^\gamma. \tag{6}$$

Therefore the components of the gradient $\boldsymbol{\nabla}\mathbf{S}$ are the partial derivatives of the components of $\mathbf{S}$. Partial derivatives are used so often that we have a special notation for them: the comma,

$$\nabla_\gamma S^\alpha{}_\beta = (\boldsymbol{\nabla}\mathbf{S})^\alpha{}_{\beta\gamma} = \frac{\partial S^\alpha{}_\beta}{\partial x^\gamma} = S^\alpha{}_{\beta,\gamma}. \tag{7}$$

C. Contraction

Contraction is an operation that takes a tensor of higher rank $\binom{M}{N}$, with $M \geq 1$ and $N \geq 1$, and generating a tensor of rank $\binom{M-1}{N-1}$. It is a generalization of the matrix trace. For definiteness, let's take a tensor $\mathbf{S}$ of rank $\binom{2}{1}$, with components $S^{\alpha\beta}{}_\gamma$. Then its contraction on the first and third indices $\mathbf{T}$ is

$$\mathbf{T}(\tilde{\boldsymbol{l}}) \equiv \sum_{\alpha=0}^{3}\mathbf{S}(\boldsymbol{\omega}^\alpha, \tilde{\boldsymbol{l}}, \boldsymbol{e}_\alpha). \tag{8}$$

(The operation of contraction can be applied to any pair of indices, one up and one down.) The components of the contraction are

$$T^\beta = \mathbf{T}(\boldsymbol{\omega}^\beta) = \sum_{\alpha=0}^{3}\mathbf{S}(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}^\beta, \boldsymbol{e}_\alpha) = \sum_{\alpha=0}^{3}S^{\alpha\beta}{}_\alpha = S^{\alpha\beta}{}_\alpha. \tag{9}$$

It can be shown (homework) that the contraction has the correct transformation properties under a change of basis.

A tensor with all indices up (or all down) can only be contracted by using the metric to lower (or raise) at least one index.

D. Divergence

The *divergence* is a calculus operation that reduces the rank of a tensor by 1: if $\mathbf{S}$ has rank $\binom{M}{N}$ then $\boldsymbol{\nabla}\cdot\mathbf{S}$ has rank $\binom{M-1}{N}$. It involves taking a gradient and then a contraction, and hence can be defined on any (upper) slot of $\mathbf{S}$. So as an example, if $\mathbf{S}$ has rank $\binom{2}{1}$ and we take the divergence on the first slot, then

$$(\boldsymbol{\nabla}\cdot\mathbf{S})(\tilde{\boldsymbol{l}}, \boldsymbol{v}) = \sum_{\alpha=0}^{3}(\boldsymbol{\nabla}\mathbf{S})(\boldsymbol{\omega}^\alpha, \tilde{\boldsymbol{l}}, \boldsymbol{v}, \boldsymbol{e}_\alpha). \tag{10}$$

In component language,

$$(\boldsymbol{\nabla}\cdot\mathbf{S})^\beta{}_\gamma = S^{\alpha\beta}{}_{\gamma,\alpha} = \frac{\partial S^{\alpha\beta}{}_\gamma}{\partial x^\alpha}. \tag{11}$$

E. Transpose

The *transpose* operation swaps input slots for a tensor, e.g. for rank $\binom{0}{2}$ tensors one can define $\mathbf{S}$ to be the transpose of $\mathbf{T}$ if

$$\mathbf{S}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{T}(\boldsymbol{v}, \boldsymbol{u}). \tag{12}$$

In components, $S_{\alpha\beta} = T_{\beta\alpha}$. This is such a simple operation that we normally don't provide another symbol for the transpose. Rather, we simply re-label the indices.

F. Symmetrization and antisymmetrization

A tensor is said to be *symmetric* if interchanging the inputs to two slots does not change it value, e.g. $\mathbf{S}(\boldsymbol{u},\boldsymbol{v}) = \mathbf{S}(\boldsymbol{v},\boldsymbol{u})$. (The metric tensor is symmetric.) Given any tensor $\mathbf{T}$, it is possible to construct a symmetric tensor $\mathbf{S}$ by linear combination with the transpose,

$$S_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) = T_{(\alpha\beta)}. \tag{13}$$

Similarly it is possible to construct an *antisymmetric* tensor,

$$A_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) = T_{[\alpha\beta]}. \tag{14}$$

These operations are encountered so frequently that they have special symbols: () for symmetrization and [] for antisymmetrization.

It is possible to symmetrize or antisymmetrize any number of indices in a tensor by permuting the indices in all possible ways and averaging:

$$T_{(\alpha_1\ldots\alpha_p)} \equiv \frac{1}{p!} \sum_{\text{all permutations } \pi \text{ of } 1\ldots p} T_{\alpha_{\pi(1)}\ldots\alpha_{\pi(p)}} \tag{15}$$

or

$$T_{[\alpha_1\ldots\alpha_p]} \equiv \frac{1}{p!} \sum_{\text{all permutations } \pi \text{ of } 1\ldots p} \pm T_{\alpha_{\pi(1)}\ldots\alpha_{\pi(p)}}, \tag{16}$$

where for antisymmetrization the $+$ or $-$ sign applies depending on whether the permutation is even or odd (even or odd number of swaps required to produce that permutation).

It is also possible to symmetrize or antisymmetrize a subset of the tensor indices.

G. Wedge product

The *wedge product* (or "exterior product") is an outer product followed by complete antisymmetrization and multiplication by the number of indices. It is used most often on vectors and 1-forms.

In the case of vectors, we define a *bivector* to be an antisymmetric tensor of rank $\binom{2}{0}$ and a trivector to be a completely antisymmetric tensor of rank $\binom{3}{0}$. Then given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we may build a bivector,

$$\boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}, \tag{17}$$

or in components

$$(\boldsymbol{u} \wedge \boldsymbol{v})^{\alpha\beta} = u^\alpha v^\beta - v^\alpha u^\beta = 2u^{[\alpha}v^{\beta]}. \tag{18}$$

One may also build a trivector $\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$,

$$(\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w})^{\alpha\beta\gamma} = 6u^{[\alpha}v^\beta w^{\gamma]}. \tag{19}$$

[Exercise: prove that the wedge product is associative.]

Beware, however: not every bivector can be written as a single term $\boldsymbol{u} \wedge \boldsymbol{v}$. Linear combinations of these are allowed.

In general, in n-dimensional spacetime, you should be able to show that a bivector has $n(n-1)/2$ independent components, and a trivector has $n(n-1)(n-2)/6$ independent components.

A similar operation holds for 1-forms. We may define a *p-form* to be a fully antisymmetric tensor of rank $\binom{0}{p}$. The wedge product of p 1-forms is a p-form. You should be able to prove that a p-form has $n!/[p!(n-p)!]$ independent components. Also, in n dimensions there is no such thing as a p-form with $p > n$ (except for the trivial zero tensor).

H. Volume tensor

The *Levi-Cevita tensor* $\boldsymbol{\epsilon}$, or volume tensor, is a special n-form (where n is the dimensionality of the spacetime) that takes in n vectors and returns the signed volume (i.e. + for right-handed and − for left-handed sets of vectors) of the parallelepiped that they span.

You learned in geometry that to find the volume of such a parallelepiped, one takes the determinant of the $n \times n$ matrix of components of the n vectors. Therefore the volume operation is indeed a tensor of rank $\binom{0}{n}$ (try explicitly proving linearity).

What are the components of the Levi-Cevita tensor? We need to find out how it acts on the basis vectors, i.e. we need $\epsilon_{\alpha_1...\alpha_n} = \boldsymbol{\epsilon}(\boldsymbol{e}_{\alpha_1}, ...\boldsymbol{e}_{\alpha_n})$. In an **orthonormal basis** $\{\boldsymbol{e}_1, ...\boldsymbol{e}_n\}$, we know that this volume is ± 1 if each of the n basis vectors appears once, with the sign depending on the handedness of the basis. The components of the Levi-Cevita tensor are then:

$$\epsilon_{\alpha_1...\alpha_n} = \begin{cases} 1 & \text{if } \alpha_1...\alpha_n \text{ is an even permutation of } 0...n-1 \\ -1 & \text{if } \alpha_1...\alpha_n \text{ is an odd permutation of } 0...n-1 \\ 0 & \text{if any indices repeated.} \end{cases} . \tag{20}$$

For the general case, we can do a transformation from an orthonormal (unprimed) basis to a general (primed) basis, $\boldsymbol{e}_{\alpha'} = L^{\beta}{}_{\alpha'}\boldsymbol{e}_{\alpha'}$. The tensor will still be fully antisymmetric so we only need to find one component, say $\epsilon_{0'...n-1'}$. For this basis transformation we find

$$\epsilon_{0'...n-1'} = L^{\alpha_1}{}'_0 ... L^{\alpha_n}{}'_{n-1} \epsilon_{\alpha_1...\alpha_n}. \tag{21}$$

This is actually an expansion in $n!$ nontrivial terms (we sum over $\alpha_1...\alpha_n$, and only the $n!$ cases where all of $1...n$ appear exactly once lead to nonzero contributions). Each of these terms is a product of n entries in $\mathbf{L}$. Inspection shows that in fact this set of terms is the determinant,

$$\epsilon_{1'...n'} = \det \mathbf{L}. \tag{22}$$

A similar operation shows that $g_{\mu'\nu'} = L^{\alpha}{}_{\mu'} L^{\beta}{}_{\nu'} \eta_{\alpha\beta}$, so that as a matrix operation

$$\mathbf{g} = -\mathbf{L}^{\mathrm{T}}\mathbf{L} \quad \rightarrow \quad \det \mathbf{L} = \sqrt{-\det \mathbf{g}} \tag{23}$$

for right-handed coordinate systems. We thus have

$$\epsilon_{\alpha'_1...\alpha'_n} = \begin{cases} \sqrt{-\det \mathbf{g}} & \text{if } \alpha'_1...\alpha'_n \text{ is an even permutation of } 0...n-1 \\ -\sqrt{-\det \mathbf{g}} & \text{if } \alpha'_1...\alpha'_n \text{ is an odd permutation of } 0...n-1 \\ 0 & \text{if any indices repeated} \end{cases} , \tag{24}$$

for right-handed coordinate systems. (For left-handed systems, we take the opposite branch of the square root.)

For Euclidean signature metrics we use δ_{ij} instead of $\eta_{\alpha\beta}$, so there is no − sign inside the square root.

I. Dual

The volume n-form enables a simple way to convert p-forms into $(n-p)$-vectors, and vice versa. We define the *dual* of a p-form $\boldsymbol{\sigma}$ to be

$$(\star\boldsymbol{\sigma})^{\alpha_1...\alpha_{n-p}} = \frac{1}{p!}\epsilon^{\beta_1...\beta_p\alpha_1...\alpha_{n-p}}\sigma_{\beta_1...\beta_p}. \tag{25}$$

The dual operation can also convert a p-vector into an $n-p$-form:

$$(\star\boldsymbol{\gamma})_{\alpha_1...\alpha_{n-p}} = \frac{1}{p!}\epsilon_{\beta_1...\beta_p\alpha_1...\alpha_{n-p}}\gamma^{\beta_1...\beta_p}. \tag{26}$$

The dual of a dual is the original object (aside from a possible minus sign depending on the dimensionality of the system).

[Beware multiple conventions for the coefficient!]

In 3 dimensions, you are probably familiar with the operation of "cross product," which kept appearing in your E&M courses. We can now see how this generalizes to any number of dimensions. Given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we may find an antisymmetric combination by taking the wedge product $\boldsymbol{u} \wedge \boldsymbol{v}$. This is now a bivector, and in 3 dimensions we can convert it to a 1-form by taking the dual:

$$\boldsymbol{\sigma} = \star(\boldsymbol{u} \wedge \boldsymbol{v}). \tag{27}$$

We can see that in 3 dimensions in an orthonormal basis, the components of $\boldsymbol{\sigma}$ are $\sigma_1 = u^2v^3 - u^3v^2$, etc. – i.e. $\boldsymbol{\sigma}$ is simply the 1-form corresponding to the vector $\boldsymbol{u} \times \boldsymbol{v}$. In more general numbers of dimensions, however, this operation does not lead to a vector, but rather a bivector or $(n-2)$-form. Operations on 2-forms play a critical role in relativistic E&M, in GR, and in gauge theories in particular physics.

J. Exterior derivative

This operation will actually be covered in detail in MTW §4, but it makes sense to introduce it here. The exterior derivative takes in a p-form $\boldsymbol{\sigma}$ and returns a $p+1$-form $\boldsymbol{d\sigma}$.

$$(\boldsymbol{d\sigma})_{\alpha_1...\alpha_{p+1}} = (p+1)\nabla_{[\alpha_1}\sigma_{\alpha_2...\alpha_{p+1}]}. \tag{28}$$

The antisymmetrization properties immediately imply that the exterior derivative of an exterior derivative is zero: $\boldsymbol{dd\sigma} = 0$ for any p-form $\boldsymbol{\sigma}$. Furthermore, in n-dimensional space, the exterior derivative of an n-form is zero.

The operation of antisymmetrized derivative is the generalization of the "curl" to an arbitrary number of dimensions. In particular, in 3 dimensions

$$\nabla \times \boldsymbol{u} = \star(\boldsymbol{du}), \tag{29}$$

where we have used the vector-1-form correspondence. However, in arbitrary numbers of dimensions, we should work with forms and multivectors.

III. ELECTRODYNAMICS IN SPECIAL RELATIVITY

[Reading: MTW §3.1, 3.3, 3.4]

We now use the tensor machinery we have built in order to write down the equations of electrodynamics.

A. Lorentz force law

The standard force law of nonrelativistic electrodynamics is that a particle of mass m and charge e experiences a force

$$m\frac{d^{(3)}\boldsymbol{v}}{dt} = e({}^{(3)}\boldsymbol{E} + {}^{(3)}\boldsymbol{v} \times {}^{(3)}\boldsymbol{B}). \quad \text{[nonrelativistic]} \tag{30}$$

In generalizing this to special relativity, one faces two difficulties: the above is not Lorentz invariant (one needs to redefine both sides of the equation to give a fully invariant form) and we haven't defined the cross product yet (what is its 4D generalization?). We can accomplish both by recalling the definition of 4-acceleration $\boldsymbol{a} = d\boldsymbol{u}/d\tau$. The electromagnetic force $m\boldsymbol{a}$ is also not a constant but must depend somehow on the electromagnetic field and the 4-velocity $\boldsymbol{u}$. The simplest option (which also happens to work!) is to write

$$\frac{dp^\alpha}{d\tau} = ma^\alpha = eF^\alpha{}_\beta u^\beta. \tag{31}$$

Here $F^\alpha{}_\beta$ is a tensor – its components form a 4×4 matrix that represents the linear operation that turns a 4-velocity into the 4-force per unit charge. Equation (31) is manifestly invariant.

[One might worry about defining a tensor this way – didn't we say a tensor took vector and 1-form inputs and return a scalar? Yes, but the operation $\mathbf{F}$ that takes a vector to a vector $\mathbf{F}(\boldsymbol{u})$ is equivalent to an operation that takes a 1-form $\boldsymbol{\sigma}$ and a vector $\boldsymbol{u}$ and returns the scalar $\langle\boldsymbol{\sigma}, \mathbf{F}(\boldsymbol{u})\rangle = F^\alpha{}_\beta\sigma_\alpha u^\beta$.]

Naively, $\mathbf{F}$ has 16 independent components, which is trouble: the familiar electromagnetic field has only 6 (3 of $\boldsymbol{E}$, 3 of $\boldsymbol{B}$). Fortunately, not all of these 16 components are truly independent. Recall that a particle's 4-velocity is normalized,

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{32}$$

Then the force law tells us that

$$0 = \frac{d}{d\tau}(g_{\alpha\beta}u^\alpha u^\beta) = 2g_{\alpha\beta}u^\alpha a^\beta = \frac{2e}{m}g_{\alpha\beta}u^\alpha F^\beta{}_\gamma u^\gamma = \frac{2e}{m}F_{\alpha\gamma}u^\alpha u^\gamma. \tag{33}$$

Since this must be true for any normalized 4-velocity $\boldsymbol{u}$, it follows that $\mathbf{F}$ must be antisymmetric. Therefore it only has 6 components.

Inspection allows us to identify these 6 components by comparison to the nonrelativistic equations. If we consider a slowly-moving particle with 3-velocity $\boldsymbol{v}$ ($|\boldsymbol{v}| \ll 1$), then to first order in $\boldsymbol{v}$ we have

$$u^0 = 1, \quad u^i = v^i. \tag{34}$$

The spatial components of the force on the particle are then

$$\frac{dp^i}{d\tau} = e(F^i{}_0 u^0 + F^i{}_j u^j) = e(F^i{}_0 + F^i{}_j v^j). \tag{35}$$

It then follows that the $F^i{}_0 = F_{i0}$ are the components of the electric field E^i, and that the $F^i{}_j = F_{ij}$ are the components of the magnetic field,

$$F_{ij} = {}^{(3)}\epsilon_{ijk}B^k, \tag{36}$$

where ${}^{(3)}\epsilon_{ijk}$ is the 3D Levi-Cevita symbol (i.e. it is +1 if $ijk = 123$, 231, or 312; -1 if $ijk = 132$, 321, or 213; and 0 otherwise). Therefore the components of $F_{\alpha\beta}$ are:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}. \tag{37}$$

As one can see, the issue involving the cross product has been resolved by making the magnetic field part of an antisymmetric tensor rather than a vector. The notion of a cross product as an antisymmetric operation on two vectors that returns another vector is inherently 3-dimensional, but antisymmetric tensors extend to any number of dimensions (including 4, which is our main concern in this class).

B. Maxwell's equations

The field equations of electrodynamics must relate the tensor $F_{\alpha\beta}$ to a source term: the 4-current density J^α. The 4-current has time component J^0 =charge density, and spatial components J^i =3-current density. One can see that these do indeed form a tensor: for a single particle traveling through spacetime along a worldline $y^i(t)$, the 4-current density is

$$J^0(t, x^i) = e\delta^{(3)}[x^i - y^i(t)] \quad \text{and} \quad J^k(t, x^i) = e\,{}^{(3)}v^k\delta^{(3)}[x^i - y^i(t)]. \tag{38}$$

It is possible [homework] to combine these as:

$$J^\alpha(x^\mu) = e\int u^\alpha \delta^{(4)}[x^\mu - y^\mu(\tau)]d\tau, \tag{39}$$

which is manifestly Lorentz-invariant.

We are now faced with the task of writing a set of equations to relate the 6 components of the field with the 4 components of the source. Since there are more field components than sources, the field $\mathbf{F}$ must satisfy some relations independent of the source. We will thus end up with two equations – one involving just the fields and the other involving the sources. These may both be derived from a single component of the equations plus Lorentz invariance.

We already know of one of these: in first-year electrodynamics we learned that $\operatorname{div}\boldsymbol{B} = 0$. Written in components,

$$\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3} = 0. \tag{40}$$

If we write this in terms of components, Eq. (37), we have

$$\frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^3} = 0. \tag{41}$$

We wrote this equation for the purely spatial components but if we are to have a Lorentz-invariant theory we need the more general equation to be true,

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0; \tag{42}$$

or, more succinctly,

$$F_{[\alpha\beta,\gamma]} = 0. \tag{43}$$

Equation (43) looks daunting but since it is explicitly antisymmetrized on 3 indices, it has $\binom{4}{3} = 4$ algebraically independent components. We have already seen one of these components, which gave div $\boldsymbol{B} = 0$. To see the others, consider the 012 component of this equation:

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0, \tag{44}$$

or

$$-\frac{\partial E^1}{\partial x^2} + \frac{\partial B^3}{\partial t} + \frac{\partial E^2}{\partial x^1} = 0. \tag{45}$$

This is the 3-component of the induction equation $\partial_t \boldsymbol{B} + \text{curl}\, \boldsymbol{E} = 0$. Relativity unifies magnetic induction with the absence of magnetic monopoles!

The other equation we need is the source equation. We start with Gauss's law, div $\boldsymbol{E} = 4\pi\rho$ (in cgs). In terms of components of $\mathbf{F}$, this says

$$\frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = 4\pi J^0. \tag{46}$$

[Recall that $F^{00} = 0$ and $F^{0i} = -F_{0i}$.] This is the 0-component of the equation

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha. \tag{47}$$

Again there are 4 nontrivial components of this equation. The spatial components give rise to Ampère's law [see homework].

In the language of forms, we may write Eqs. (43) and (47) as

$$\boldsymbol{dF} = 0 \quad \text{and} \quad -\boldsymbol{d}\star\boldsymbol{F} = 4\pi\star\boldsymbol{J}. \tag{48}$$

The first of these is trivial to show, the second takes some work.

C. Automatic conservation of the source

The attentive reader will note that Maxwell's equations give us 8 differential equations for 6 variables. Therefore they are either redundant or overconstrained. It turns out that there is 1 redundant equation and 1 overconstraint.

The 1 redundant equation comes from Eq. (43): if we define $C_{\alpha\beta\gamma} = F_{[\alpha\beta,\gamma]}$ then it follows from commuting partial derivatives that

$$C_{[\alpha\beta\gamma,\delta]} = 0. \tag{49}$$

This is a single equation [it has $\binom{4}{4} = 1$ component] that is simply true mathematically for any $\mathbf{F}$, so in fact there are only 3 fully independent equations in Eq. (43).

There is also a single overconstraint. Let's take the divergence of Eq. (47):

$$F^{\alpha\beta}{}_{,\beta\alpha} = 4\pi J^\alpha{}_{,\alpha}. \tag{50}$$

Since $\mathbf{F}$ is antisymmetric, but the ordering of partial derivatives is irrelevant, the left-hand side of this equation is zero (expanded in components, $F^{12}{}_{,21}$ will cancel against $F^{21}{}_{,12}$). Therefore

$$J^\alpha{}_{,\alpha} = 0. \tag{51}$$

This is the law of *conservation of charge*, as one can see by writing it in components:

$$\frac{\partial\rho}{\partial t} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0. \tag{52}$$

The conservation of charge is "automatic" in the sense of being implied by the field equations: if charge is not conserved there is no solution!