

## Ph 236 – Homework 8

Due: Wednesday, December 6, 2012

*Note: This will be a half-homework.*

### 1. Variational approach to universes with matter. [18 points]

This problem works through the Lagrangian derivation of the Friedmann equations, and then considers some issues associated with the “total energy of the Universe” in cosmology. When we consider the variational principle, we will focus only on variations that respect the FRW symmetry (inhomogeneous universes will be considered later).

We will show next term that the action for general relativity is the Einstein-Hilbert action,

$$S_{\text{EH}} = \int \frac{R}{16\pi} \sqrt{-g} d^4x, \quad (1)$$

where  $g$  denotes the determinant of the metric tensor, and  $R$  is the Ricci scalar. Recall that  $\sqrt{-g} d^4x$  is the differential proper 4-volume element.

(b) Using the Ricci scalar from the notes, show that for an FRW universe, if we consider the comoving volume  $\mathcal{V}$ , then the action may be written as an integral containing  $\ddot{a}$ . Reduce this to a form containing at most first time derivatives of  $a$  using integration by parts and throwing out surface terms to show that

$$S_{\text{EH}} = \frac{3}{8\pi} \mathcal{V} \int [-a\dot{a}^2 + aK] dt. \quad (2)$$

(c) Recall that the action for a point particle of mass  $\mu$  is  $S_{\text{part}} = -\mu \int d\tau$ , where  $d\tau$  is the differential of proper time. Show that if these particles are at rest in the comoving frame, and there is a comoving density  $\rho_0$ , then this action reduces to

$$S_{\text{part}} = -\rho_0 \mathcal{V} \int dt. \quad (3)$$

We therefore conclude that in a universe with cold noninteracting particles, the total action is

$$S = \mathcal{V} \int \left[ -\frac{3}{8\pi} a\dot{a}^2 + \frac{3K}{8\pi} a - \rho_0 \right] dt. \quad (4)$$

(d) Show that if this action is varied with fixed  $t_1$ ,  $t_2$ ,  $a(t_1)$ , and  $a(t_2)$ , allowing the trajectory of  $a$  to vary in between the initial and final times, that the Euler-Lagrange equation gives the **second-order equation**:

$$\frac{\ddot{a}}{a} = -\frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 - \frac{K}{2a^2}. \quad (5)$$

Show that the Friedmann equations imply Eq. (5), **but** that Eq. (5) does not imply the Friedmann equations.

You will note that in part (d) we did not allow  $t_1$  or  $t_2$  to vary. In general relativity – unlike Newtonian physics or even special relativity – the action should be stationary with respect to variations where we keep the initial and final states fixed but allow arbitrary behavior in between, including changing the total proper time seen by the matter particles, which in the FRW coordinates is  $t_2 - t_1$ . Therefore, in part (d), we did not allow the most general legal variation. We should have fixed the initial and final  $a$  and allowed  $t_1$  and  $t_2$  to float.

(e) Re-write Eq. (4) in terms of the function  $t(a)$ . Allowing general variations with  $t_1$  and  $t_2$  free but fixing  $a_1$  and  $a_2$ , show that you can derive an equation involving  $t(a)$ . Re-writing it in terms of  $a(t)$ , show that you arrive at the **first-order equation**:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi\rho_0}{3a^3} + \frac{K}{a^2}. \quad (6)$$

This is an example of a *constraint equation* – a constraint on initial conditions (i.e. on a field and its first derivative) due to the existence of gauge degrees of freedom (in this case choosing the labeling of surfaces of constant  $t$ ). The fact that we have fixed the gauge by setting  $g_{tt} = -1$  does not remove the constraint!

(f) In any system of the form  $S = \int L(a, \dot{a}) dt$ , where  $L$  is a Lagrangian with no explicit time dependence, it is possible to construct a conserved Hamiltonian  $H$ . In ordinary mechanics (including special relativity),  $H$  is identified with the total energy. Construct  $H$  for the FRW universe and show that it is always equal to zero.