Ph 236 – Homework 7

Due: Wednesday, November 28, 2012

1. Gravitomagnetic precession. [18 points]

In class, I mentioned that objects with angular momentum create a gravitomagnetic field around them described by the potential $\bar{h}_{0i}$. In this problem, you will evaluate the effect of this field on satellites in a polar orbit around a rotating body (such as the Earth).

You may assume that the Earth has mass $M_\oplus$, radius $R_\oplus$, and rotates at angular velocity $\Omega_\oplus$ around the 3-axis. The real Earth is not uniform density (the core is denser) but for this problem you may assume uniform density.

In all of the responses below, you may compute metric potentials and perturbations outside the Earth.

(a) Evaluate the angular momentum of the Earth, and then the potential components $\bar{h}_{00}$ and $\bar{h}_{0i}$ as a function of spatial position.

(b) Evaluate the Christoffel symbols $\Gamma^i{}_{00}$ and $\Gamma^i{}_{0j}$.

(c) Show that the acceleration of an object relative to the coordinate system, i.e. d^2x^i/dt^2, is of the form

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\Phi + \frac{d\boldsymbol{x}}{dt} \times \boldsymbol{B}. \tag{1}$$

What is the gravitational potential Φ and the gravitomagnetic field $\boldsymbol{B}$?

We will imagine that there is a satellite that orbits in the 13-plane on a circular orbit of radius a. The satellite has mass $\mu \ll M_\odot$ (so that it can be treated as a test particle). We are now going to find the Newtonian angular momentum of the *satellite*, and the torque on the satellite's orbit due to the rotation of the Earth – then it should be possible to find the precession rate.

(d) Find the angular velocity $\omega = 2\pi/T$ of the satellite's orbit and its angular momentum $\boldsymbol{L}$.

(e) The additional acceleration computed in (c) corresponds to an effective force on the satellite. Compute the torque and average it over the orbit, $\langle\boldsymbol{\Gamma}\rangle$.

(f) Based on your computed torque, show that the orientation of the satellite's orbit precesses. Express the precession rate in terms of $M_\oplus$, $R_\oplus$, $\Omega_\oplus$, and a. Is this in the same or opposite direction as the Earth's rotation?

(g) Numerically evaluate the answer to (f) for a low-orbiting (400 km altitude) satellite around the Earth. What would the precession rate be around a millisecond pulsar? [You can look up typical values in your favorite reference.]

2. Gravitational wave emission from a binary star. [18 points]

Consider a binary star with masses M_1 and M_2. They orbit around their common center of mass, with the relative separation vector $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ describing an ellipse with semimajor axis a and eccentricity e. This problem works through the effect of gravitational radiation on the orbit, and ultimately you will learn whether the orbit circularizes or becomes more eccentric as it shrinks. You may assume the orbit is in the 12-plane without loss of generality. Also the orbit is large enough that standard Newtonian gravity should be usable to describe it (aside from the long-term evolution – perihelion precession and gravitational wave inspiral).

Each part of the problem may be worked to the lowest relevant order in e, as denoted in brackets. (This is to simplify the algebra: the final answer to the problem has an exact solution that is an analytic function of e, even though some of the intermediate steps do not!) [You will need the quadrupole power emission and angular momentum emission formulae, which are in MTW §36.7 and will be covered in class before the due date.]

(a) How do the mean motion n (defined as a frequency by $2\pi/T$ where T is the orbital period), orbital energy E, and angular momentum L relate to a and e? [To order e^2.]

(b) Suppose that the periastron occurs on the 1-axis, i.e. where $\boldsymbol{r} = (a(1-e), 0, 0)$, at time $t = 0$. As a function of time for this Keplerian orbit (you may consider the period $0 \le t < T$, so that Keplerian formulae may be used), what is the quadrupole tensor Q_{ij}? [To order e^2.]

(c) Now find the emitted power $-\dot{E}$ and angular momentum $-\dot{L}$. [To order e^2.]

(d) Using the formulae from (a), convert $\dot{E}$ and $\dot{L}$ from (c) to find $\dot{a}$ and $\dot{e}$. [To order e^2 for $\dot{a}$ and e^1 for $\dot{e}$.] Show that the orbit becomes more circular as it shrinks.