Ph 236 – Homework 6

Due: Wednesday, November 14, 2012

1. Cosmological constant. [16 points]

Consider the highly symmetrical spacetime with line element given by

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2), \tag{1}$$

where $H > 0$ is a constant, and in the domain $\eta < 0$. (This is known as *de Sitter spacetime*, and H is the Hubble constant.) [Note: if the "dark energy" really is a cosmological constant, our distant future will be well-approximated by this spacetime.]

(a) Find the Einstein tensor for this spacetime.

(b) Prove that de Sitter spacetime is a vacuum solution of Einstein's equations in the presence of a positive cosmological constant for a particular choice of H. What is the relation between H and Λ?

(c) Consider an observer whose world line is given by fixed spatial coordinates (x, y, z). Explain why this trajectory is a geodesic, and show that an infinite amount of proper time elapses before the observer reaches $\eta = 0$.

2. Conservation laws. [20 points]

A vector field $\boldsymbol{\xi}$ is called a *Killing field* if $\xi_{(\alpha;\beta)} = 0$.

(a) Recall from class that under an infinitesimal change of coordinates given by a vector $\boldsymbol{\xi}$ that the metric changed by

$$\Delta g_{\mu\nu} = -\xi^\alpha g_{\mu\nu,\alpha} - \xi^\alpha{}_{,\mu} g_{\alpha\nu} - \xi^\alpha{}_{,\nu} g_{\alpha\mu}. \tag{2}$$

Show that this is equal to zero if and only if $\boldsymbol{\xi}$ is a Killing field. Thus a Killing field describes a symmetry of the spacetime.

(b) Consider a particle following along a geodesic with some 4-momentum $p^\alpha = dx^\alpha/d\lambda$. Show that if $\boldsymbol{\xi}$ is a Killing field, the inner product $K = \boldsymbol{p} \cdot \boldsymbol{\xi}$ is conserved along the trajectory.

(c) Show that if $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ are Killing fields, then also the commutator $\boldsymbol{\chi} = [\boldsymbol{\xi}, \boldsymbol{\psi}]$ is a Killing field. [*Hint*: Explicitly expand the covariant derivative $\chi_{(\alpha;\beta)}$, and use the Riemann tensor.]

(d) Find the Killing fields associated with spatial translations and rotations in Minkowski space. What are the corresponding conservation laws?